Math 471 Final Exam and Real Analysis Prelim

December 19, 2018

Instructions: Please write legibly. You may do the problems in any order. If you use more than one blue book, write your name on each, also indicate on the covers which problems are in which. Provide complete proofs, and justifications for calculations. You may cite a result from class or the text, fully stating what you are using, **unless it is, or is clearly equivalent to, the result you are being asked to prove**. The first five problems, designated by (P), form the Real Analysis Prelim.

Unless otherwise indicated, $[a, b]$ refers to a general closed interval in $\mathbb{R}$ and $\overline{\mathbb{R}}$ is the extended real numbers; *measurable* refers to Lebesgue measurable subsets of $\mathbb{R}^d$ (the collection of all of which is $\mathcal{M}(\mathbb{R}^d)$) or measurable functions; $m(E)$ denotes the Lebesgue measure of an $E \in \mathcal{M}$; the dimension d is general unless specified; functions are assumed to be $\overline{\mathbb{R}}$-valued unless otherwise stated; and *integrable* means Lebesgue integrable.

1 (P). For an integrable $f : [0, \infty) \to \overline{\mathbb{R}}$, define the *Laplace transform* of f by the formal expression

$$F(s) = \int_{\mathbb{R}_+} e^{-sx} f(x)dx,\ 0 < s < \infty.$$

(i) Show that if f is integrable on $[0, \infty)$, one has $F \in L^\infty(0, \infty)$.

(ii) Show that if $(1 + |x|)f(x)$ is integrable on $[0, \infty)$, then F is differentiable on $(0, \infty)$, and find $F'(s)$.

2 (P). On $\mathbb{R}$, let

$$K(x) = -1,\ -1 \leq x < 0; = +1,\ 0 \leq x \leq 1;\ \text{ and } = 0,\ |x| > 1.$$

For $0 < \delta < \infty$, let $K_\delta(x) = \delta^{-1}K(\frac{x}{\delta})$, and let $*$ denote convolution. Prove that for $f \in L^1(\mathbb{R})$, $f * K_\delta \to 0$ in L^1 as $\delta \to 0^+$.

3 (P). State Hölder's inequality for measurable functions on $\mathbb{R}^d$, and prove it, assuming Young's inequality for real numbers.

4 (P). Let $E \in \mathcal{M}(\mathbb{R}^d)$ with $m(E) > 0$, and let $(L^1(E), ||\cdot||_{L^1})$ be the Banach space of integrable functions on E. Define $weak - L^1(E) := wk - L^1(E)$ to be the space of measurable f on E such that there exists a $B < \infty$ s.t.

$$(*) \quad (\forall \lambda > 0)\, m\{x \in E : |f(x)| > \lambda\} \leq \frac{B}{\lambda},$$

and for $f \in wk - L^1$, let $||f||_{wk-L^1}$ be the infimum of all B such that $(*)$ holds.

(i) Prove that $wk - L^1(E)$ is a vector space (with respect to pointwise addition and scalar multiplication of functions).

(ii) Prove that $L^1(E) \subsetneq wk - L^1(E)$.

For $0 < \delta < \infty$, define $f_\delta(x) = \delta^{-1} f(\frac{x}{\delta})$. Prove that for all measurable f and all δ,

(iii) $||f_\delta||_{L^1} = ||f||_{L^1}$ and

(iv) $||f_\delta||_{wk-L^1} = ||f||_{wk-L^1}$.

5 (P). Let $f : [a, b] \to \mathbb{R}$ be a function.

(i) Define what it means for f to be (a) Lipschitz continuous; (b) absolutely continuous (AC) ; and (c) of bounded variation (BV).

(ii) Prove that AC implies BV but that the converse is false.

(iii) We know by a theorem in class and the textbook that if $F : [a, b] \to \mathbb{R}$ is AC, then F is differentiable a.e., and $F' \in L^1[a, b]$. Is the converse true? Prove or give a counterexample.

6. Let $E \in \mathcal{M}(\mathbb{R})$, $E \subset [0, \frac{1}{2}]$, and suppose that for a.e. $x \in E$,

$$(**) \quad x - r \leq \frac{m\Big(E \cap [x - r, x + r]\Big)}{m([x - r, x + r])} \leq x + r, \text{ a.e. } r,\, 0 < r < x.$$

What can you say about E? Hint: consider what happens for x a Lebesgue point of density of E.