Math 471 Final Exam - Analysis Prelim

December 20, 2017

Instructions: Please write legibly. You may do the problems in any order. If you use more than one blue book, write your name on each, also indicate on the covers which problems are in which. Provide complete proofs, and justifications for calculations. You may cite a result from class or the text, fully stating what you are using, **unless it is, or is clearly equivalent to, the result you are being asked to prove**. The five problems which form the Real Analysis Prelim are designated by (P).

Unless otherwise indicated, $[a, b]$ refers to a general closed interval in $\mathbb{R}$, *measurable* refers to Lebesgue measurable subsets of $\mathbb{R}$ (the collection of all of which is $\mathcal{M}$) and $\mathbb{R}^*$-valued functions on $\mathbb{R}$, and *integrable* means Lebesgue integrable. On a general measurable space, a *measure* means a nonnegative measure. For a set X, $2^X = \mathcal{P}(X)$ denotes the power set of X.

1 (P). Let $f : [a, b] \to \mathbb{R}$.

(i) Define what it means for f to be of *bounded variation.*

(ii) Define what it means for f to be *Lipschitz.*

(iii) Prove that if f is Lipschitz, then it is of bounded variation.

(iv) Prove that the converse is not true.

2 (P). (i) Define what it means for a function $g : \mathbb{R} \to \mathbb{R}^*$ to be *measurable.*

(ii) Prove that if $g : \mathbb{R} \to \mathbb{R}$ is measurable and $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h \circ g$ is measurable.

3. Suppose that $F : \mathbb{R} \to \mathbb{R}^*_+$ is measurable and e^F is integrable on $\mathbb{R}$.

(i) Prove that $F \in L^p(\mathbb{R})$ for every $1 \le p < \infty$.

(ii) Suppose that F is supported in $[0, 1]$. Prove that

$$\int_0^1 F\,dx \le \log\left(\int_0^1 e^F\,dx\right).$$

4 (P). Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$. Define a function $\mu^* : 2^{\mathbb{R}} \to \mathbb{R}$ by

$$\mu^*(E) = \sum_{n \in E,\, n=1}^{\infty} a_n.$$

(i) Prove that μ^* is an outer measure.

(ii) Determine the collection $\mathcal{N} \subset 2^{\mathbb{R}}$ of subsets which are measurable with respect to μ^*.

5 (P). Suppose that $(X, \mathcal{M}, \mu)$ is a (nonnegative) measure space.

(i) For $1 \leq p < \infty$, define $L^p(X, \mu)$.

(ii) Prove that if $\mu(X) < \infty$, then for every pair, $1 \leq p \leq q < \infty$, $L^q(X, \mu) \subseteq L^p(X, \mu)$.

(iii) If $(X, \mathcal{M}, \mu)$ is such that $L^q(X, \mu) \subseteq L^p(X, \mu)$ for all $1 \leq p \leq q < \infty$, does it follow that $\mu(X) < \infty$?

6 (P). Let $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ be sequences of nonnegative functions in $L^2(\mathbb{R})$ with $||f_n||_{L^2} \leq n^{-2/3}$ and $||g_n||_{L^2} \leq n^{-1/2}$ for all n. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of all rational numbers $r_n \in \mathbb{Q}$. Prove that the series

$$\sum_{n=1}^{\infty} f_n(x - r_n) g_n(x + r_n)$$

converges for a.e. $x \in \mathbb{R}$.