\[ \cong \int_{D \times D} \mathcal{D}(A \oplus B, D) \otimes \mathcal{D}(D_1, A) \otimes \mathcal{D}(D_2, B) \]
\[ \cong \int_{\mathcal{D} \times \mathcal{D}} \mathcal{D}(D_1, A) \otimes \mathcal{D}(D_2, B) \otimes \mathcal{D}(A \oplus B, D) \]
\[ \cong \mathcal{C}((D_1, D_2), (A, B)) \otimes F((A, B)) \]
\[ \text{where } \mathcal{C} := \mathcal{D} \times \mathcal{D} \text{ and } F((A, B)) := \mathcal{D}(A \oplus B, D) \]
\[ \cong F((D_1, D_2)) \quad \text{by Proposition 3.2.25} \]
\[ \cong \mathcal{D}(D_1 \oplus D_2, D) - (\mathcal{D}_D \mathcal{D}_D)_D. \]

The following is proved by Mandell et al in [MMSS01, 22.1] in the case of topological categories.

**Proposition 3.3.15.** Lax symmetric monoidal functors and commutative algebras. The category of (commutative) monoids in \([\mathcal{D}, \mathcal{V}]\) is isomorphic to that of lax (symmetric) monoidal functors \(\mathcal{D} \rightarrow \mathcal{V}\) (Definition 2.6.19).

**Proof.** Let \(R : \mathcal{D} \rightarrow \mathcal{V}\) be lax (symmetric) monoidal. Then, in the notation of Definition 2.6.19, we have a unit map \(\iota : 1 \rightarrow R(0)\) and a natural transformation \(\mu : R(\cdot) \otimes R(\cdot) \rightarrow R(\cdot \oplus \cdot)\). By the definition of the tensored Yoneda functor \(F^0\) and the Yoneda functor \(1 \rightarrow \mathcal{V}\) of Yoneda Lemma 2.2.10, the maps \(\iota\) and \(\mu\) determine and are determined by the maps \(\eta : 1 \rightarrow R\) and \(m : R \oplus R \rightarrow R\) of Definition 2.6.58 that give \(R\) the structure of a (commutative) monoid. \(\square\)

### 3.4 Simplicial sets and simplicial spaces

The category of simplicial sets is a convenient combinatorial substitute for that of topological spaces and a widely used tool in homotopy theory. A thorough modern account can be found in [GJ99].

#### 3.4A The category of finite ordered sets

Let \(\Delta\) be the category of finite ordered sets \([n] - \{0, 1, \ldots, n\}\) and order preserving maps. It is an easy exercise to show that any such map can be written as a composite of the following ones:

- the **face maps** \(d_i : [n - 1] \rightarrow [n]\) for \(0 \leq i \leq n\), where \(d_i\) is the order preserving monomorphism that does not have \(i\) in its image and
- the **degeneracy maps** \(s_i : [n + 1] \rightarrow [n]\) for \(0 \leq i \leq n\), where \(s_i\) is the order preserving epimorphism sending \(i\) and \(i + 1\) to \(i\).

These satisfy the **simplicial identities**:
(i) $d_i d_j = d_{i-1} d_j$ for $i < j$
(ii) $d_i s_j = s_{j-1} d_i$ for $i < j$
(iii) $d_i s_j = id$ for $i = j$ and for $i = j + 1$
(iv) $d_j s_j = s_1 d_{j-1}$ for $i > j + 1$
(v) $s_i s_j = s_j s_{i-1}$ for $i > j$.

**Definition 3.4.1.** A simplicial set $X$ is a functor $\Delta^{op} \to \mathcal{S}et$. It is common to denote its value on $[n]$ by $X_n$ and call it the set of $n$-simplices of $X$. A simplicial set $X$ thus consists of a collection of sets $X_n$ for $n \geq 0$, along with face maps $d_i : X_n \to X_{n-1}$ and degeneracy maps $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$ satisfying the identities (i)–(v) above. A simplex is nondegenerate if it is not in the image of any degeneracy map $s_i$. The category $\mathcal{S}et_\Delta$ of simplicial sets is the category of such functors with natural transformations as morphisms.

More generally a simplicial object $X$ in a category $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$. It is common to write it as $X_\bullet$, to emphasize its simplicial nature. We denote the category of simplicial objects in $\mathcal{C}$ by $\mathcal{C}_\Delta$.

Similarly a cosimplicial object $Y$ in a category $\mathcal{C}$, sometimes denoted by $Y^\bullet$, is a $\mathcal{C}$ valued functor on $\Delta$ whose value on $[n]$ is denoted by $Y^n$. It consists of a collection of objects $Y^n$ in $\mathcal{C}$ for $n \geq 0$, along with coface maps $d^i : Y^{n-1} \to Y^n$ and codegeneracy maps $s^i : Y^{n+1} \to Y^n$ for $0 \leq i \leq n$ satisfying identities dual to (i)–(v) above. We denote the category of cosimplicial objects in $\mathcal{C}$ by $\mathcal{C}^\Delta$. In particular, a cosimplicial space is an object in the category $\mathcal{Top}^{\Delta}$ of functors $\Delta \to \mathcal{Top}$.

For an object $C$ in $\mathcal{C}$, we denote by $cc_u(C)$ the constant simplicial object at $C$, the functor $\Delta^{op} \to \mathcal{C}$ sending each object to $C$ and each morphism to $1_C$. The constant cosimplicial object at $C$, $cc_c(X)$ is similarly defined.

Simplicial sets are ubiquitous in homotopy theory, but cosimplicial sets are rarely considered. Cosimplicial spaces are more common.

**Definition 3.4.2.** The cosimplicial space $\Delta^*$, the cosimplicial standard simplex, is the functor $[n] \mapsto \Delta^n$, where the standard $n$-simplex $\Delta^n$ is the space

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0 \text{ and } \sum_i t_i = 1 \right\}.$$ 

It is homeomorphic to the $n$-disk $D^n$. Its boundary $\partial \Delta^n$ is the set of points with at least one coordinate equal to 0; it is homeomorphic to $S^{n-1}$. The $i$th face $\Delta^n_i$ for $0 \leq i \leq n$ is the set of points with $t_i = 0$; it is homeomorphic to $D^{n-1}$. The $i$th horn $\Lambda^n_i$ is the complement of the interior of the $i$th face in the boundary, the set of points with at least one vanishing coordinate and with $t_i > 0$. It is also homeomorphic to $D^{n-1}$.

It is an inner horn if $0 < i < n$; otherwise it is an outer horn.
The cosimplicial standard simplicial set $\Delta[\bullet]$ (called the cosimplicial standard simplex in [Hir03, Definition 15.1.15]) is the functor $[n] \mapsto \Delta[n]$, where the simplicial set $\Delta[n]$ (also called the standard $n$-simplex) is given by

$$\Delta[n]_k = \Delta([k],[n]).$$

The singular chain complex for $Y$ is obtained from the free abelian groups on these sets by defining a boundary operator in terms of the face maps $d_i$.

Definition 3.4.3. The geometric realization $|X|$ (or $\mathcal{R}e(X)$) of a simplicial set $X$ is the coend (Definition 2.4.5)

$$|X| := \int \Delta X_n \times \Delta^n.$$

This means the topological space $|X|$ is the quotient of the union of all of the simplices of $X$,

$$\bigsqcup_n X_n \times \Delta^n,$$

obtained by gluing them together appropriately. Equivalently it is the quotient of a similar disjoint union using only the nondegenerate simplices of $X$. In particular the space $\Delta^n$ is $|\Delta[n]|$ for the simplicial set $\Delta[n]$ of Definition 3.4.2.

The geometric realization $|X|$ of a simplicial space $X$ is similarly defined as a quotient of the union of the spaces $X_n \times \Delta^n$, whose topologies are determined by those of the spaces $X_n$ as well the spaces $\Delta^n$.

Remark 3.4.4. Following common practice, we are using the term “standard $n$-simplex” for both the topological space $\Delta^n$ and the simplicial set $\Delta[n]$ of Definition 3.4.2 in hopes that the distinction between the two will be clear from the context. Note that $|\Delta[n]| \cong \Delta^n$, so $|\Delta[\bullet]| \cong \Delta^\bullet$.

Remark 3.4.5. The realization of a bisimplicial set. It follows from the definitions that the coend

$$\int \Delta X_n \times \Delta[n]$$

is the simplicial set $X$ itself. Now suppose that $X$ is a bisimplicial set, meaning a simplicial object in the category of simplicial sets or equivalently set valued functor on $\Delta^{op} \times \Delta^{op}$. Then in the coend above, each $X_n$ is itself a simplicial set, and the coend is another simplicial set $|X|$. Hirschhorn [Hir03, Definition 15.11.1] calls this the realization of the bisimplicial set $X$. In [Hir03, Theorem 15.11.6] he shows that it is naturally isomorphic to the diagonal simplicial set

$$\Delta^{op} \xrightarrow{\text{diag}} \Delta^{op} \times \Delta^{op} \xrightarrow{X} \mathcal{S}et.$$ (3.4.6)
Definition 3.4.7. The singular functor. For a topological space $Y$ the simplicial set $\text{Sing}(Y)$ (the singular complex of $Y$) is given by letting $\text{Sing}(Y)_n$ be the set of all continuous maps $\Delta^n \to Y$. The face and degeneracy operators are defined in terms of the coface and codegeneracy operators on $\Delta$.

The following is proved by May in [May67, 14.1].

Proposition 3.4.8. $|X|$ as a CW complex. The geometric realization $|X|$ of a simplicial set $X$ is a CW complex with one $n$-cell for each nondegenerate $n$-simplex of $X$.

Similarly we have a map

$$\coprod_n X_n \to \bigwedge \Delta X_n,$$

which is the set $\pi_0 |X|$ of path connected components of $|X|$. Thus collapsing each $\Delta^n$ to a point in Definition 3.4.3 gives a map

$$|X| - \bigwedge \Delta^n X_n \xrightarrow{e} \bigwedge \Delta X_n = \pi_0 |X|.$$  \hspace{1cm} (3.4.9)

A simplicial space $X$, i.e., a functor $X : \Delta^{op} \to \textbf{T}_{\text{op}}$, has a geometric realization $|X|$ defined as in Definition 3.4.3, but with the not necessarily discrete topology of $X_n$ taken into account.

For a simplicial set $X$, $|X[n]|$ is the $n$-skeleton of the CW complex $|X|$.

The following was proved by Kan in [Kan58a].

Proposition 3.4.10. The equivalence of $\textbf{Set}_\Delta$ and $\textbf{T}_{\text{op}}$ and of their pointed analogs. As a functor from $\textbf{Set}_\Delta$ to $\textbf{T}_{\text{op}}$, geometric realization of Definition 3.4.3 is the left adjoint of $\text{Sing}$, the singular functor of Definition 3.4.7. The adjunction $|\cdot| : \textbf{Set}_\Delta \rightleftarrows \textbf{T}_{\text{op}} : \text{Sing}$ and its pointed analog are equivalences of categories.

In particular for an arbitrary space $X$ one has a weak homotopy equivalence $|\text{Sing}(X)| \to X$ whose source is a CW complex. For this reason, e.g., in [BK72] (the “yellow monster”), the terms “space” and “simplicial set” are sometimes used interchangeably.

Definition 3.4.11. Topological and simplicial categories.

(i) When $\mathcal{V} = (\textbf{T}_{\text{op}}, \times, *)$, we say that a $\mathcal{V}$-category is a topological category. We denote the category of topological categories by $\text{CAT}_{\text{TOP}}$ and that of small topological categories by $\text{Cat}_{\text{TOP}}$.

(ii) When $\mathcal{V} = (\mathcal{T}, \land, S^0)$, we say that a $\mathcal{V}$-category is a pointed topological category. We denote the category of pointed topological categories by $\text{CAT}_T$ and that of small pointed topological categories by $\text{Cat}_T$. 

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(iii) When $\mathcal{V} = (\text{Set}_\Delta \times \ast, \ast)$, we say that a $\mathcal{V}$-category is a simplicial category. We denote the category of simplicial categories by $\text{CAT}_\Delta$ and that of small simplicial categories by $\text{Cat}_\Delta$.

(iv) When $\mathcal{V} = (\text{Set}_\Delta \times \ast, S^0)$, we say that a $\mathcal{V}$-category is a pointed simplicial category. We denote the category of simplicial categories by $\text{CAT}_\Delta$ and that of small pointed simplicial categories by $\text{Cat}_\Delta$.

We will see below in Corollary 5.6.16 that every topological model category is also a simplicial one.

The adjunction

$$| \cdot | : \text{Set}_\Delta \xrightarrow{\sim} \text{Top} : \text{Sing}$$

leads to

$$| \cdot | : \text{CAT}_\Delta \xrightarrow{\sim} \text{CAT}_{\text{Top}} : \text{Sing}$$

(see Definition 3.4.11) in the obvious way. Given a simplicial category $\mathcal{C}$, we define the topological category $|\mathcal{C}|$ to have the same objects as $\mathcal{C}$ with morphism spaces

$$|\mathcal{C}|(X,Y) = |\mathcal{C}(X,Y)|,$$

and given a topological category $\mathcal{D}$, we define the simplicial category $\text{Sing}(\mathcal{D})$ to have the same objects as $\mathcal{D}$ with simplicial morphisms sets

$$\text{Sing}(\mathcal{D})(X,Y) = \text{Sing}(\mathcal{D}(X,Y)).$$

3.4B The nerve of a small category

Definition 3.4.12. The nerve and classifying space of a small (topological) category. For a small category $J$, the nerve $N(J)$ is the simplicial set given by

$$N(J)_n = \text{Cat}([n], J)$$

where $[n]$ here denotes the linearly ordered set $\{0, \ldots, n\}$ regarded as a category. The classifying space $BJ$ is the geometric realization of the nerve, $|N(J)|$.

For a small topological category $\mathcal{D}$, the similarly defined nerve $N(\mathcal{D})$ is a simplicial space whose geometric realization (see Definition 3.4.3) is the classifying space $B\mathcal{D}$.

In other words, $N(J)_n$ is the set of diagrams in $J$ of the form

$$j_0 \to j_1 \to \cdots \to j_{n-1} \to j_n.$$  \hfill (3.4.13)

Of the $n + 1$ face maps $N(J)_n \to N(J)_{n-1}$, $n - 1$ are obtained by composing each of the $n-1$ pairs of adjacent arrows above, and the other two are obtained