DEFINITION. LET $\mathcal{C}$ AND $\mathcal{D}$ BE CATEGORIES

A **FUNCTOR** $F : \mathcal{C} \to \mathcal{D}$ ASSIGNS

1) TO EACH OBJECT X IN $\mathcal{C}$ AN OBJECT $F(X)$ IN $\mathcal{D}$
2) FOR EACH MORPHISM $X \xrightarrow{\alpha} Y$ IN $\mathcal{C}$ A MORPHISM

$$F(X) \xrightarrow{F(\alpha)} F(Y) \quad \text{IN } \mathcal{D}$$

SUCH THAT FOR $W \xrightarrow{\alpha} X \xrightarrow{\beta} Y$ IN $\mathcal{C}$

WE GET $\quad F(W) \xrightarrow{F(\alpha)} F(X) \xrightarrow{F(\beta)} F(Y)$ IN $\mathcal{D}$

WITH $F(\beta)F(\alpha) = F(\beta\alpha)$.

EXAMPLES ① $\mathcal{C}$ = CATEGORY OF POINTED TOP. SPACES (X, x_0) AND POINTED CONTINUOUS MAPS.

$\mathcal{D}$ = CATEGORY OF GROUPS

WE HAVE DEFINED THE FUNDAMENTAL GROUP $\pi_1(X, x_0)$ FOR EACH OBJECT

GROUP $\pi_1(X, x_0)$ FOR EACH OBJECT OF $\mathcal{C}$. THUS π_1 IS A FUNCTOR

$Top_* = \mathcal{C} \longrightarrow \mathcal{D} = Grp$.

(2) RECALL FOR A GROUP G, BG IS THE ONE OBJECT CATEGORY WITH A MORPHISM $\forall \gamma \in G$ AND COMPOSITION GIVEN BY GROUP MULTIPLICATION. LET $X : BG \to Top$. IT DEFINES A SPACE X WITH AN ACTION OF G, i.e. $\forall \gamma \in G$ WE A MAP $X \xrightarrow{\gamma} X$, NECESSARILY A HOMEOMORPHISM. THE FUNCTOR DEFINES A HOMOMORPHISM FROM G TO THE HOMEOMORPHISM GROUP OF X.

(3) $Top \xrightarrow{U} Set$ FORGETFUL FUNCTOR

Top = CAT OF TOP SPACES

Set = CAT. OF SETS

CAT OF TOP. SPACES   CAT. OF SETS

(4) $\mathcal{C} =$

$$\begin{tikzcd} \text{JOHN} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \text{PAUL} \arrow[d, "\gamma"] \\ \text{GEORGE} \arrow[r, "\delta"] & \text{RINGO} \end{tikzcd}$$

WITH $\gamma\alpha = \delta\beta$

$\mathcal{D} = \mathcal{S}ets$

A FUNCTOR $\mathcal{C} \xrightarrow{F} \mathcal{D}$ IS

A DIAGRAM

$$\begin{tikzcd} F(\text{JOHN}) \arrow[r, "F(\alpha)"] \arrow[d, "F(\beta)"'] & F(\text{PAUL}) \arrow[d, "F(\gamma)"] \\ F(\text{GEORGE}) \arrow[r, "F(\delta)"] & F(\text{RINGO}) \end{tikzcd}$$

IN $\mathcal{S}et$

<u>DEF</u> AN <u>INITIAL / TERMINAL OBJECT</u> I IN $\mathcal{C}$ IS ONE FROM / TO WHICH THERE IS EXACTLY ONE MORPHISM TO / FROM EVERY OTHER OBJECT.

<u>EASY FACT</u> ANY 2 INITIAL OBJECTS ARE ISOMORPHIC.

<u>DEFINITION</u> A CATEGORY IS

DEFINITION A CATEGORY IS **POINTED** IF EACH INITIAL OBJECT IS TERMINAL AND VICE VERSA.

EXAMPLE $\mathcal{T}op_*$, THE CATEGORY OF POINTED SPACES, IS POINTED.

ADJOINT FUNCTORS

$F \dashv U$

KAN TURNSTILE

$$X \in \mathcal{C} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{D} \ni Y$$

F IS THE LEFT ADJOINT OF U

U IS THE RIGHT ADJ. OF F

THESE FUNCTORS ARE **ADJOINT** IF

$$\mathrm{Hom}_{\mathcal{C}}(X, UY) \xrightarrow[\sim]{\Phi_{X,Y}} \mathrm{Hom}_{\mathcal{D}}(FX, Y)$$

$\mathrm{Hom}_{\mathcal{C}}(X, UY)$ = SET OF MORPHISMS IN $\mathcal{C}$ $X \to UY$

$\mathrm{Hom}_{\mathcal{D}}(FX, Y)$ = SET OF MORPHISMS IN $\mathcal{D}$ $FX \to Y$

BIJECTION

WHICH IS NATURAL IN BOTH X AND Y (LATER)

EXAMPLE

EXAMPLE

$\mathcal{C} = \mathcal{S}et \qquad \mathcal{D} = \mathcal{A}b =$ CATEGORY OF ABELIAN GROUPS

F = FREE ABELIAN GROUP FUNCTOR

U = FORGETFUL FUNCTOR

$$X \in \mathcal{S}et \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{A}b \ni A$$

$$\mathrm{Hom}_{\mathcal{S}et}(X, UA) \overset{?}{\cong} \mathrm{Hom}_{\mathcal{A}b}(FX, A)$$

YES

$$\begin{tikzcd} X \in \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \ni Y \arrow[l, shift left, "U"] \end{tikzcd}$$

$$\mathrm{Hom}_{\mathcal{C}}(X, UY) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(FX, Y)$$

① SUPPOSE $Y = FX$

$$\mathrm{Hom}_{\mathcal{C}}(X, UFX) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(FX, FX)$$

\eta

$X \xrightarrow{\eta_X} UFX$ UNIT OF THE

$1_{\ldots}$

\eta $\quad X \xrightarrow{\eta_X} UFX$ UNIT OF THE ADJUNCTION $\quad 1_{FX}$

(2) SUPPOSE $\quad X = UY$

$$\mathrm{Hom}_{\mathcal{C}}(UY, UY) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FUY, Y)$$

$$1_{UY} \longmapsto \varepsilon_Y : FUY \to Y$$

COUNIT OF ADJUNCTION

$$X \in \mathcal{S}et \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{A}b \ni A$$

(1) $A = FX =$ FREE AB GP ON X

WE GET A MAP $X \longrightarrow UFX$

$x \longmapsto$ GENERATOR OF FX CORRESPONDING TO x

(2) $X = UA$

$FUA \longrightarrow A$

"LINEAR ''

"LINEAR
EXTENSION"
OF LA