FOR PATHS $p, q : (I, \partial I) \to (X, x_0)$

$$(p * q)(t) := \begin{cases} p(2t) & \text{FOR } 0 \leq t \leq 1/2 \\ q(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

IF $p \simeq p'$ HOMOTOPIC TO

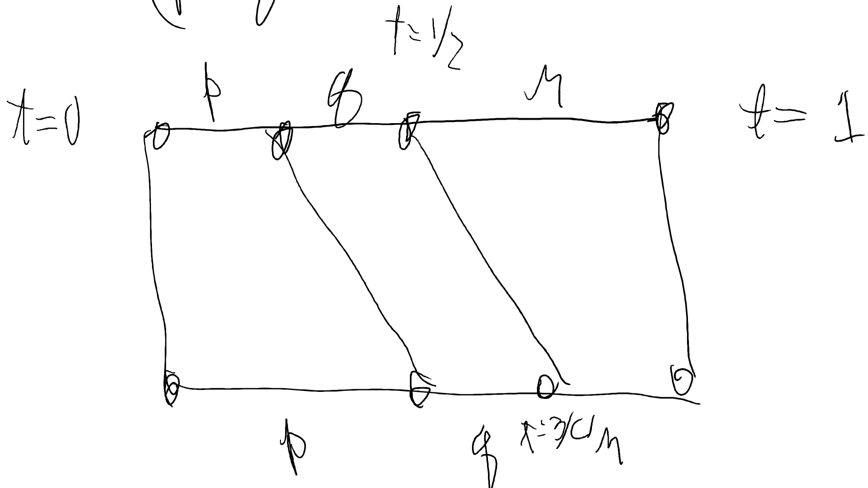
AND $q \simeq q'$, THEN $p * q \simeq p' * q'$

WE HAVE A BINARY OPERATION ON THE SET OF EQUIV. CLASSES

① ASSOCIATIVITY

LET p, q, r BE CLOSED PATHS

COMPARE $(p * q) * r$ AND $p * (q * r)$



$(p * q) * r$

$p * (q * r)$

$p \qquad q \; x=y\,r$

CAN SHOW $(p * q) * r \simeq p * (q * r)$

② LET e BE THE CONSTANT PATH. CAN SHOW THAT

$$p * e \simeq e * p \simeq p$$

③ INVERSES. DEFINE $\bar{p}$ (THE INVERSE OF p) BY $\bar{p}(t) = p(1-t)$.

CAN SHOW

$$p * \bar{p} \simeq e \simeq \bar{p} * p.$$

QED

THIS GROUP $\pi_1(X, x_0)$ NEED <u>NOT</u> BE ABELIAN

<u>NICE PROPERTY</u>:

SUPPOSE WE HAVE

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

$(X, x_0) \longrightarrow (Y, y_0)$

WE GET A GROUP HOM.

$$\alpha \in \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

IS REP'D BY

$$(I, \partial I) \xrightarrow{a} (X, x_0) \xrightarrow{f} (Y, y_0)$$

REMARK: IF X IS PATH CONNECTED, ANY CHOICE OF x_0 LEADS TO THE SAME GROUP.

SOME FACTS TO BE PROVEN LATER

1. $\pi_1(\mathbb{R}^n, 0) = 0 \quad \forall n$ EASY
2. $\pi_1(S^1) \cong \mathbb{Z}$ HARD

$$S^n := \{ (x_0, \dots x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1 \}$$

$= n$-SPHERE

3. $SO(2) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \leq \theta \leq 2\pi \right\}$

$$\left\{ \begin{bmatrix} \cdots & \cdots \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \leq \theta \leq 2\pi \right\}$$

$$\approx S^1$$

Ex $\pi_1(SO(3)) \cong \mathbb{Z}/2$ AND THE HOMOMORPHISM

$$\begin{array}{ccc} \pi_1 SO(2) & \longrightarrow & \pi_1 SO(3) \\ \| & & \| \\ \mathbb{Z} & & \mathbb{Z}/2 \end{array}$$

IS ONTO.

CATEGORY THEORY

DEF A <u>CATEGORY</u> CONSISTS OF

a) A COLLECTION OF OBJECTS

b) FOR ANY TWO OBJECTS X, Y THERE IS A SET OF MORPHISMS $X \longrightarrow Y$

SUCH THAT

1) MORPHISMS

i) MORPHISMS

$$X \xrightarrow{f} Y \text{ AND } Y \xrightarrow{g} Z$$

CAN BE COMPOSED

$$X \xrightarrow{gf} Z$$

ii) COMPOSITION IS ASSOCIATIVE

iii) EACH X HAS AN IDENTITY MORPHISM $X \xrightarrow{1_X} X$

EXAMPLES

1. SETS AND MAPS
2. TOP. SPACES AND CONTINUOUS MAPS
3. GROUPS AND HOMOMORPHISMS
4. LET G BE A GROUP
   $\mathcal{B}G$ = CATEGORY WITH ONE OBJECT AND A MORPHISM $\forall\, g \in G$

   WHERE COMPOSITION IS GROUP MULTIPLICATION

MULTIPLICATION

5. CATEGORY WITH 4 OBJECTS JOHN, PAUL, GEORGE + RINGO

$$\begin{tikzcd} \text{JOHN} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \text{PAUL} \arrow[d, "\beta"] \\ \text{GEORGE} \arrow[r, "\delta"] & \text{RINGO} \end{tikzcd}$$

WITH $\delta\gamma = \beta\alpha$

LET $\mathcal{C}$ AND $\mathcal{D}$ BE CATEGORIES AND <u>FUNCTOR</u> $F : \mathcal{C} \to \mathcal{D}$ CONSIST OF

1) $\forall$ OB

TO BE CONTINUED.