Math 443 Final exam

December 15, 2016

Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. (30 POINTS) **Three dimensional grid question.** Let $X_0 \subset \mathbf{R}^3$ be the set of points (x, y, z) in which two of the three coordinates are integers. It is an infinite union of lines parallel to the coordinate axes, and each point with three integer coordinates is the intersection of three such lines.

   Choose a number ϵ with $0 < \epsilon < 1/2$ and let $X_1 \subset \mathbf{R}^3$ be the set of points whose distance from X_0 is $\leq \epsilon$. It is a 3-manifold whose boundary X_2 is a noncompact surface.

   Let $G = \mathbf{Z}^3 \subset \mathbf{R}^3$. It acts freely on all three spaces by translation, and each orbit space is compact.

   (a) Decscribe the graph X_0/G and compute its Euler characteristic and fundamental group.

   (b) Find the genus of the surface X_2/G.

   HINT: Consider the intersection of each X_i with the cube $[-1/2, 1/2]^3$. Then $\mathbf{R}^3/G$ is the 3-torus obtained by making appropriate identifications on this cube. The orbit spaces X_0/G and X_2/G can be studied in a similar way.

   Solution:

   (a) X_0/G has one vertex, the image of $(0, 0, 0)$, and three edges, the images of the three coordinate axes. Hence its Euler characteristic is -2 and its fundamental group is free on 3 generators.

   (b) X_1/G is the 3-manifold obtained by attaching three handles to D^3 and its boundary X_2/G has genus 3.

2. (20 POINTS) **Torus question.** Prove or disprove the Borsuk-Ulam theorem for the torus, which says the following. For every map $f : S^1 \times S^1 \to \mathbf{R}^2$, there is a point (x, y) in $S^1 \times S^1$ such that $f(-x, -y) = f(x, y)$. Here we regard S^1 as the unit circle in the complex numbers in order to define $-z$ for $z \in S^1$.

   Solution: The theorem is false. Let f be the composite of projection p_1 onto the first coordinate followed by an embedding i of S^1 into the plane. Then we have

   $$f(-x, -y) = i(-x) \neq i(x) = f(x, y).$$

3. (20 POINTS) **Nonplanar graph question.** Let K be the houses and utilities graph. It has six vertices, x_1, x_2, x_3, y_1, y_2, and y_3. Each x_i is connected to each y_j by an edge, so there are nine edges. Use an Euler characteristic argument to prove that K cannot be embedded in the plane. HINT: SHOW THAT EACH FACE MUST BE BOUNDED BY AT LEAST 4 EDGES.

> **Solution:** The vertices of a face must be alternately houses and utilities since each edge connects a house to a utility. Hence each face has an even number of edges. The number cannot be two because we cannot have two edges connecting the same house to the same utility, so it must be at least four.
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> A spherical polyhedron with 6 vertices and 9 edges must have 5 faces in order to have Euler characteristic 2. The hint implies that $E \geq 2F$ since each edge belongs to 2 faces. This is a contradiction.

4. (20 POINTS) **The two cell CW complex question.** Let X be the quotient of the unit disk in the complex numbers $\mathbf{C}$ obtained by identifying each point z on the boundary with ζz, where $\zeta = e^{2\pi i/5}$. Find $\pi_1 X$ and prove your answer.

> **Solution:** Let $A \subseteq X$ be the closed disk of radius 1/2 centered at 0, and let B be the complement of its interior in X. Then $A \cap B = S^1$, A is contractible, and B is homotopy equivalent to a circle. The inclusion of $A \cap B$ into B is a map of degree 5. Thus the van Kampen diagram is
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> $$\begin{tikzcd} & & 0 \arrow[d, equal] \\ & & \pi_1 A \\ \mathbf{Z} \arrow[r, equal] \arrow[uurr] \arrow[ddrr, "5"'] & \pi_1 A \cap B \arrow[ur] \arrow[dr] & \\ & & \pi_1 B \arrow[d, equal] \\ & & \mathbf{Z} \end{tikzcd}$$
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> The pushout group is $\mathbf{Z}/5$.

5. (20 POINTS) **Covering space question.** Let $\zeta = e^{2\pi i/3}$, let $\tilde{X}$ be the complement of the set

$$\left\{z_0 = 0, z_1 = 1, z_2 = \zeta, z_3 = \zeta^2\right\}$$

in $\mathbf{C}$ (the complex numbers), and let X be the complement of the set $\{0, 1\}$ in $\mathbf{C}$. Let $p : \tilde{X} \to X$ be defined by $p(z) = z^3$. Using the point $\tilde{x}_0 = 1/2 \in \tilde{X}$ as a base point, we define four closed paths ω_k for $0 \le k \le 3$ in $\tilde{X}$ as follows:

$$\begin{aligned}
\omega_0(t) &= e^{2\pi it}/2 && \text{for } 0 \le t \le 1 \\
\omega_1(t) &= 1 - (e^{2\pi it}/2) && \text{for } 0 \le t \le 1 \\
\omega_2(t) &= \begin{cases} e^{2\pi it}/2 & \text{for } 0 \le t \le 1/3 \\ \zeta(1 - (e^{6\pi it}/2)) & \text{for } 1/3 \le t \le 2/3 \\ e^{-2\pi it}/2 & \text{for } 2/3 \le t \le 1 \end{cases} \\
\omega_3(t) &= \begin{cases} e^{-2\pi it}/2 & \text{for } 0 \le t \le 1/3 \\ \zeta^2(1 - (e^{6\pi it}/2)) & \text{for } 1/3 \le t \le 2/3 \\ e^{2\pi it}/2 & \text{for } 2/3 \le t \le 1 \end{cases}
\end{aligned}$$

(I suggest you draw a picture of these paths.)

(a) (5 POINTS) Find $\pi_1(\tilde{X}, \tilde{x}_0)$ and describe the elements in it represented by the 4 closed paths ω_k.

> **Solution:** Since $\tilde{X}$ is the complement of 4 points in the plane, its π_1 is the free group on 4 generators, say a_k for $0 \le k \le 3$. The four paths each go around one of them in a counterclockwise direction, so each ω_k represents one of the generators a_k.

(b) (5 POINTS) Show that p is a 3-sheeted covering.

> **Solution:** The preimage of every every point in X is a set of three points in $\tilde{X}$.

(c) (5 POINTS) Let $x_0 = p(\tilde{x}_0) \in X$ and find $\pi_1(X, x_0)$. Describe the elements in it represented by the 4 closed paths $p\omega_k$. You may assume that the image under p of a circle of radius 1/2 about a cube root of unity is a simple closed curve going counterclockwise around 1 and not going around 0.

> **Solution:** Since X is the complement of 2 points in the plane, its π_1 is the free group on 2 generators, say x and y corresponding to 0 and 1. Then drawing suitable pictures shows that
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> $$\begin{aligned} p(a_0) &= x^3 \\ p(a_1) &= y \\ p(a_2) &= xyx^{-1} \\ p(a_3) &= x^{-1}yx \end{aligned}$$

(d) (5 POINTS) Find a homomorphism $\varphi : \pi_1(X, x_0) \to C_3$ whose kernel contains $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

> **Solution:** Let $\gamma \in C_3$ be a generator, and define φ by $\varphi(x) = \gamma$ and $\varphi(y) = e$.