Math 443 — Final exam

December 15, 2014

Be sure to write your name on your bluebook. Use a separate page (or pages) for each problem. Show all of your work.

1. (20 POINTS) Consider the subset V_d of the complex projective plane $\mathbf{C}P^2$ defined by the equation

$$x^d + y^d + z^d = 0 \qquad \text{for a positive integer } d.$$

It is known as the FERMAT CURVE of degree d. Define a map $f : V_d \to \mathbf{C}P^1$ by

$$[x, y, z] \mapsto [x, y].$$

A map of this type is called a BRANCHED COVERING. It does *not* extend to all of $\mathbf{C}P^2$ because it is not defined on the point $[0, 0, 1]$.

(a) Find and count the points in the target whose preimage is *not* a set of d points in V_d. Let $K \subseteq \mathbf{C}P^1$ denote the set of these points. They are called BRANCH POINTS.

(b) You may assume that the restriction of f to the preimage of $\mathbf{C}P^1 - K$ is a d-fold covering of $\mathbf{C}P^1 - K$. Use this fact to find the Euler characteristic of V_d. You may also use the fact that under suitable hypotheses, $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Solution:

(a) The preimage of $[x, y]$ is the set

$$\left\{[x, y, z] \colon z^d = -x^d - y^d\right\}$$

There are d such values of z unless $x^d + y^d = 0$. There are d such points in $\mathbf{C}P^1$, namely

$$\left\{[1, -e^{2\pi ik/d}] \colon 0 \le k < d\right\},$$

so K has d points.

(b) The Euler characteristic of $\mathbf{C}P^1 - K$ is $2 - d$, so that of its preimage is $2d - d^2$. The preimage of d small disks around the points of K is d. It follows that $\chi(V_d) = d + 2d - d^2 = 3d - d^2$.

2. (20 POINTS) Let K be the complete graph on six vertices, meaning that there is a single edge connecting each pair of vertices. Use an Euler characteristic argument to prove that K cannot be embedded in the plane.

Solution: A face cannot be bounded by just two edges, because they would have to connect the same pair of vertices. Hence every face has at least 3 edges, so $E \geq 3F/2$, where E and F denote the number of edges and faces.

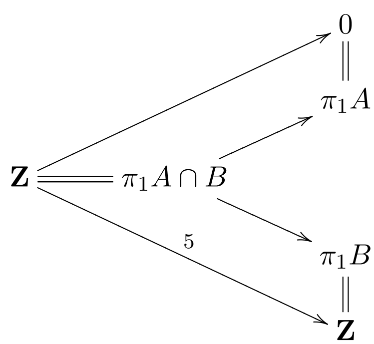
A spherical polyhedron with 6 vertices and 15 edges must have 11 faces in order to have Euler characteristic 2. Since $E \geq 3F/2$, this is a contradiction.

3. (20 POINTS) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk D^2 to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbf{Z}$.

Solution: See page 32 of Hatcher.

4. (20 POINTS) Let X be the quotient of the unit disk in the complex numbers $\mathbf{C}$ obtained by identifying each point z on the boundary with ζz, where $\zeta = e^{2\pi i/5}$. Find $\pi_1 X$ and prove your answer.

Solution: Let $A \subseteq X$ be the closed disk of radius 1/2 centered at 0, and let B be the complement of its interior in X. Then $A \cap B = S^1$, A is contractible, and B is homotopy equivalent to a circle. The inclusion of $A \cap B$ into B is a map of degree 5. Thus the van Kampen diagram is



The pushout group is $\mathbf{Z}/5$.

5. (20 POINTS)

(a) Describe the space X of the previous problem as a CW-complex and find the homology of its cellular chain complex.

(b) Use the Künneth Theorem to find $H_*(X \times X)$ and $H_*(X \times \mathbf{R}P^2)$.

Solution: (a) X has a single cell in dimensions 0, 1 and 2. The 1-skeleton is a circle and the 2-cell is attached by a map of degree 5. Hence the cellular chain complex is

$$\begin{array}{ccccc} C_0 & \xleftarrow{0} & C_1 & \xleftarrow{5} & C_2 \\ \| & & \| & & \| \\ \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z} \end{array}$$

and its homology is

$$H_i(X) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ \mathbf{Z}/5 & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Note that $\mathrm{Tor}_1(\mathbf{Z}/5, \mathbf{Z}/5) = \mathbf{Z}/5$ and $\mathrm{Tor}_1(\mathbf{Z}/5, \mathbf{Z}/2) = 0$. It follows that

$$H_i(X \times X) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ \mathbf{Z}/5 \oplus \mathbf{Z}/5 & \text{for } i = 1 \\ \mathbf{Z}/5 \otimes \mathbf{Z}/5 = \mathbf{Z}/5 & \text{for } i = 2 \\ \mathrm{Tor}_1(\mathbf{Z}/5, \mathbf{Z}/5) = \mathbf{Z}/5 & \text{for } i = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(X \times \mathbf{R}P^2) = \begin{cases} \mathbf{Z} & \text{for } i = 0 \\ \mathbf{Z}/5 \oplus \mathbf{Z}/2 = \mathbf{Z}/10 & \text{for } i = 1 \\ \mathbf{Z}/5 \otimes \mathbf{Z}/2 = 0 & \text{for } i = 2 \\ \mathrm{Tor}_1(\mathbf{Z}/5, \mathbf{Z}/2) = 0 & \text{for } i = 3 \\ 0 & \text{otherwise.} \end{cases}$$