Math 443 **Final exam**

December 19, 2022

Name:

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ____

Problems begin below, and there are two blank pages to write your answer on following each of three problems.

1. **Infinite graph question.** (40 POINTS.) Consider the infinite graph K in $\mathbf{R}^3$ with vertex set
$$\{(i,j,k) \in \mathbf{R}^3 \colon i,j,k \in \mathbf{Z}\} \cup \left\{\left(\frac{2i+1}{2}, \frac{2j+1}{2}, \frac{2k+1}{2}\right) \in \mathbf{R}^3 \colon i,j,k \in \mathbf{Z}\right\}$$
in which each vertex of the form (x,y,z) is connected by an edge to the eight neighboring vertices
$$\left\{\left(x \pm \frac{1}{2}, y \pm \frac{1}{2}, y \pm \frac{1}{2}\right)\right\}.$$
Thus the center of each edge is a point in the set
$$\left\{\left(i \pm \frac{1}{4}, j \pm \frac{1}{4}, k \pm \frac{1}{4}\right) : i,j,k \in \mathbf{Z}\right\}.$$
The two endpoints for such an edge with a given combination of signs are
$$(i,j,k) \qquad \text{and} \qquad \left(i \pm \frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2}\right)$$
with the same combination of signs in the second point.

Let L be the set of points within ϵ of K, for some positive $\epsilon < 1/4$. It is a noncompact compact 3-manifold with boundary in $\mathbf{R}^3$. Its boundary M is a noncompact surface.

The group $G = \mathbf{Z}^3$ acts freely $\mathbf{R}^3$ by translation, with $(i,j,k) \in \mathbf{Z}^3$ sending $(x,y,z) \in \mathbf{R}^3$ to $(x+i, y+j, z+k)$. Hence it acts freely on both K and M. *Describe the finite orbit graph K/G and find the genus of the compact orbit surface M/G.* Both K/G and M/G are contained in the 3-dimensional torus $\mathbf{R}^3/G \cong S^1 \times S^1 \times S^1$, which is also a quotient of the unit cube.

Workspace for problem 1 continued.

Workspace for problem 1 continued.

2. **Complete bipartite graph question.** (30 POINTS.) A *bipartite graph* is one in which the vertices fall into two disjoint sets, say red and blue vertices, and each edge connects a red vertex to a blue one. It is *complete* if there is a unique edge connecting each red vertex to each blue one.

   Let $K_{m,n}$ denote the complete bipartite graph with m red vertices and n blue ones. Hence it has mn edges.

   Show that if $K_{m,n}$ can be embedded in a closed oriented surface of genus g, then

$$g \geq \frac{(m-2)(n-2)}{4}.$$

   In particular, $g > 0$, so the graph is nonplanar, for $m = n = 3$. $K_{3,3}$ is known as the houses and utilities graph.

Workspace for problem 2 continued.

Workspace for problem 2 continued.

3. (30 POINTS) **Covering space question.** Let $\zeta = e^{2\pi i/3}$, let $\tilde{X}$ be the complement of the set

$$\{z_0 = 0, z_1 = 1, z_2 = \zeta, z_3 = \zeta^2\}$$

in $\mathbf{C}$ (the complex numbers), and let X be the complement of the set $\{0, 1\}$ in $\mathbf{C}$. Let $p : \tilde{X} \to X$ be defined by $p(z) = z^3$. Using the point $\tilde{x}_0 = 1/2 \in \tilde{X}$ as a base point, we define four closed paths ω_k for $0 \le k \le 3$ in $\tilde{X}$ as follows:

$$\begin{aligned}
\omega_0(t) &= e^{2\pi i t}/2 && \text{for } 0 \le t \le 1 \\
\omega_1(t) &= 1 - (e^{2\pi i t}/2) && \text{for } 0 \le t \le 1 \\
\omega_2(t) &= \begin{cases} e^{2\pi i t}/2 & \text{for } 0 \le t \le 1/3 \\ \zeta(1 - (e^{6\pi i t}/2)) & \text{for } 1/3 \le t \le 2/3 \\ e^{-2\pi i t}/2 & \text{for } 2/3 \le t \le 1 \end{cases} \\
\omega_3(t) &= \begin{cases} e^{-2\pi i t}/2 & \text{for } 0 \le t \le 1/3 \\ \zeta^2(1 - (e^{6\pi i t}/2)) & \text{for } 1/3 \le t \le 2/3 \\ e^{2\pi i t}/2 & \text{for } 2/3 \le t \le 1 \end{cases}
\end{aligned}$$

(I suggest you draw a picture of these paths.)

(a) (5 POINTS) Find $\pi_1(\tilde{X}, \tilde{x}_0)$ and describe the elements in it represented by the 4 closed paths ω_k.

(b) (5 POINTS) Show that p is a 3-sheeted covering.

(c) (5 POINTS) Let $x_0 = p(\tilde{x}_0) \in X$ and find $\pi_1(X, x_0)$. Describe the elements in it represented by the 4 closed paths $p\omega_k$. You may assume that the image under p of a circle of radius 1/2 about a cube root of unity is a simple closed curve going counterclockwise around 1 and not going around 0.

(d) (5 POINTS) Find a homomorphism $\varphi : \pi_1(X, x_0) \to C_3$ whose kernel contains $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

Workspace for problem 3 continued.

Workspace for problem 3 continued.