Math 443

Final exam
December 13, 2021

Name:

Pledge of Honesty
I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ________________________________________________________________

Problems begin below, and there are two blank pages to write your answer on following each of five problems.

1. 3-manifold homology question.  (30 points.) Find the homology of the 3-manifold obtained by “attaching $k$ handles” to the 3-sphere $S^3$. “Attaching a handle” to a 3-manifold means $M$ the following:

   - Remove two disjoint open disks from $M$, thus obtaining a manifold $M'$ bounded by two copies of $S^2$.
   - The cylinder $S^2 \times I$ is another 3-manifold with the same boundary.
   - Form a new closed 3-manifold $N$ by identifying the boundaries of $M'$ and $S^2 \times I$.

One can use the Mayer-Vietoris sequence to compute $H_* M'$ in terms of $H_* M$, and $H_* N$ in terms of $H_* M'$. You can assume that $H_3 X = 0$ for a connected 3-manifold with boundary $X$, and that $H_3 Y = \mathbb{Z}$ for a connected 3-manifold without boundary $Y$.

Starting with $S^3$, do the above $k$ times to obtain a 3-manifold $M_k$. Equivalently, one could remove $2k$ disjoint open disks from $S^3$ and identify the resulting boundary with that of $k$ copies of $S^2 \times I$.

Note that the 2-dimensional analog of this process leads from $S^2$ to a surface of genus $k$. 

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Workspace for problem 1 continued.
Workspace for problem 1 continued.
2. **Infinite graph question.** (30 points.) Consider the infinite graph \( K \) in \( \mathbb{R}^3 \) with vertex set

\[
\{(i, j, k) \in \mathbb{R}^3 : i, j, k \in \mathbb{Z}\} \cup \left\{ \left( \frac{2i+1}{2}, \frac{2j+1}{2}, \frac{2k+1}{2} \right) : i, j, k \in \mathbb{Z} \right\}
\]

in which each vertex of the form \((x, y, z)\) is connected by an edge to the eight neighboring vertices

\[
\left\{ \left( x \pm \frac{1}{2}, y \pm \frac{1}{2}, z \pm \frac{1}{2} \right) \right\}.
\]

Thus the center of each edge is a point in the set

\[
\left\{ \left( i \pm \frac{1}{4}, j \pm \frac{1}{4}, k \pm \frac{1}{4} \right) : i, j, k \in \mathbb{Z} \right\}.
\]

The two endpoints for such an edge with a given combination of signs are

\[
(i, j, k) \quad \text{and} \quad \left( i \pm \frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2} \right)
\]

with the same combination of signs in the second point.

Let \( L \) be the set of points within \( \epsilon \) of \( K \), for some positive \( \epsilon < 1/4 \). It is a noncompact compact 3-manifold with boundary in \( \mathbb{R}^3 \). Its boundary \( M \) is a noncompact surface.

The group \( G = \mathbb{Z}^3 \) acts freely \( \mathbb{R}^3 \) by translation, with \((i, j, k) \in \mathbb{Z}^3 \) sending \((x, y, z) \in \mathbb{R}^3 \) to \((x+i, y+j, z+k)\). Hence it acts freely on both \( K \) and \( M \). Describe the finite orbit graph \( K/G \) and find the genus of the compact orbit surface \( M/G \). Both \( K/G \) and \( M/G \) are contained in the 3-dimensional torus \( \mathbb{R}^3/G \cong S^1 \times S^1 \times S^1 \), which is also a quotient of the unit cube.
Workspace for problem 2 continued.
Workspace for problem 2 continued.
3. **Euler characteristic question.** (20 points.) Let $X$ be a finite graph with $V$ vertices and $E$ edges. Embed it in $\mathbb{R}^3$ (there is a theorem saying that any graph can be embedded in 3-space; there are some that cannot be embedded in the plane) and let $Y$ be the space of all points within $\epsilon$ (a sufficiently small positive number) of the image of $X$. It is a 3-manifold bounded by a surface $M$. Find the Euler characteristic $\chi(M)$ and prove your answer.

**Hint:** Think of the building set in the lounge, the one with steel balls and black magnetic rods. We are going to build something with $V$ balls and $E$ rods. Find the Euler characteristic of the set of $V$ 2-spheres bounding the $V$ balls. Think about how the Euler characteristic of the surface changes each time you add a rod. *You may use the fact that*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

*under suitable hypotheses on $A$ and $B*. 
Workspace for problem 3 continued.
Workspace for problem 3 continued.
4. **Complete bipartite graph question.** (20 POINTS.) A *bipartite graph* is one in which the vertices fall into two disjoint sets, say red and blue vertices, and each edge connects a red vertex to a blue one. It is *complete* if there is a unique edge connecting each red vertex to each blue one.

Let $K_{m,n}$ denote the complete bipartite graph with $m$ red vertices and $n$ blue ones. Hence it has $mn$ edges.

Show that if $K_{m,n}$ can be embedded in a closed oriented surface of genus $g$, then

$$g \geq \frac{(m - 2)(n - 2)}{4}.$$ 

In particular, $g > 0$, so the graph is nonplanar, for $m = n = 3$. $K_{3,3}$ is known as the houses and utilities graph.
Workspace for problem 4 continued.
Workspace for problem 4 continued.
5. **Brouwer Fixed Point question.** (20 points) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk $D^2$ to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbb{Z}$. 
Workspace for problem 5 continued.
Workspace for problem 5 continued.