1. Fermat curve question. (20 points) Consider the subset $V_d$ of the complex projective plane $\mathbb{C}P^2$ defined by the equation

$$x^d + y^d + z^d = 0$$

for a positive integer $d$.

It is known as the Fermat curve of degree $d$. Define a map $f : V_d \rightarrow \mathbb{C}P^1$ by

$$[x, y, z] \mapsto [x, y].$$

A map of this type is called a branched covering. It does not extend to all of $\mathbb{C}P^2$ because it is not defined on the point $[0, 0, 1]$.

(a) Find and count the points in the target whose preimage is not a set of $d$ points in $V_d$. Let $K \subseteq \mathbb{C}P^1$ denote the set of these points. They are called branch points.

(b) You may assume that the restriction of $f$ to the preimage of $\mathbb{C}P^1 - K$ is a $d$-fold covering of $\mathbb{C}P^1 - K$. Use this fact to find the Euler characteristic of $V_d$. You may also use the fact that under suitable hypotheses, $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.

2. Complete bipartite graph question. 20 POINTS A bipartite graph is one in which the vertices fall into two disjoint sets, say red and blue vertices, and each edge connects a red vertex to a blue one. It is complete if there is a unique edge connecting each red vertex to each blue one.

Let $K_{m,n}$ denote the complete bipartite graph with $m$ red vertices and $n$ blue ones. Hence it has $mn$ edges. For example, $K_{3,3}$ is the houses and utilities graph, which is known to be nonplanar.

Show that if $K_{m,n}$ can be embedded in a closed oriented surface of genus $g$, then

$$g \geq \frac{(m - 2)(n - 2)}{4}.$$
3. **Five lemma question.** (20 POINTS) The 5-lemma says that given a commutative diagram of abelian groups with exact rows,

\[
\begin{array}{cccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\epsilon} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E'
\end{array}
\]

if \(\alpha, \beta, \delta\) and \(\epsilon\) are isomorphisms, then so is \(\gamma\). Show by counterexample that the triviality of \(\alpha, \beta, \delta\) and \(\epsilon\) does not imply the triviality of \(\gamma\).

4. **Brouwer Fixed Point question.** (20 POINTS) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk \(D^2\) to itself has a fixed point. You may assume \(\pi_1 S^1 = \mathbb{Z}\).

5. **James reduced product question.** (20 POINTS) Let the 2-sphere \(S^2\) have base point \(x_0\). Define an equivalence relation on the \(n\)-fold Cartesian product \((S^2)^\times n\) by saying that when a coordinate in a point

\[(x_1, x_2, \ldots, x_n) \in (S^2)^\times n\]

is the base point \(x_0\), it may be transposed with either the coordinate on its left or the one its right. For example the points

\[(x_0, x_1, x_2), (x_1, x_0, x_2)\] and \((x_1, x_2, x_0) \in (S^2)^\times 3\]

are all equivalent for arbitrary points \(x_1, x_2 \in S^2\). The space \(J_n S^2\) of equivalence classes in \((S^2)^\times n\) is called the \(n\)th James reduced product of \(S^2\), having first been studied by Ioan James in the 1950s. One could make a similar definition with \(S^2\) replaced by any pointed space. Thus there is a surjective map \(f_n : (S^2)^\times n \to J_n S^2\).

We know that \((S^2)^n\) has a CW-structure with \(\binom{n}{k}\) cells in dimension \(2k\) for \(0 \leq k \leq n\). We also know that as a ring under cup product,

\[H^*((S^2)^\times n; \mathbb{Z}) \cong \mathbb{Z}[y_i : 1 \leq i \leq n]/(y_i^2),\]

with \(y_i \in H^2\) being the generator associated with the \(i\)th factor of the Cartesian product.

It can be shown that \(J_n S^2\) has a CW-structure with a single cell in every even dimension up to \(2n\). For \(1 \leq k \leq n\), the group \(H^{2k}(J_n S^2; \mathbb{Z}) \cong \mathbb{Z}\) has a generator \(u_k\) whose image under \(f_n^*\) is the sum of all square free \(k\)-fold products of the \(y_i\)s.

Use this information to determine the cup product structure of \(H^*(J_n S^2; \mathbb{Z})\). Give a formula for \(u_k u_\ell\) as a multiple of \(u_{k+\ell}\) for \(k + \ell \leq n\). In particular \(f_n\) induces a monomorphism in cohomology.

**Hint:** Try doing this first for small values of \(n\) such as 2, 3 and 4.