Math 443 **Final exam**

December 14, 2020

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

SCAN THIS PAGE WITH THE HONOR PLEDGE SIGNEDAND UPLOAD IT WITH YOUR EXAM

Use a separate page (or pages) for each problem. Show all of your work.

1. **Fermat curve question.** (20 POINTS) Consider the subset V_d of the complex projective plane $\mathbf{C}P^2$ defined by the equation

$$x^d + y^d + z^d = 0 \qquad \text{for a positive integer } d.$$

It is known as the *Fermat curve* of degree d. Define a map $f: V_d \to \mathbf{C}P^1$ by

$$[x, y, z] \mapsto [x, y].$$

A map of this type is called a *branched covering*. It does *not* extend to all of $\mathbf{C}P^2$ because it is not defined on the point $[0, 0, 1]$.

(a) Find and count the points in the target whose preimage is *not* a set of d points in V_d. Let $K \subseteq \mathbf{C}P^1$ denote the set of these points. They are called BRANCH POINTS.

(b) You may assume that the restriction of f to the preimage of $\mathbf{C}P^1 - K$ is a d-fold covering of $\mathbf{C}P^1 - K$. Use this fact to find the Euler characteristic of V_d. You may also use the fact that under suitable hypotheses, $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Solution:

(a) The preimage of $[x, y]$ is the set

$$\left\{[x, y, z] \colon z^d = -x^d - y^d\right\}$$

There are d such values of z unless $x^d + y^d = 0$. There are d such points in $\mathbf{C}P^1$, namely

$$\left\{[1, -e^{2\pi ik/d}] \colon 0 \leq k < d\right\},$$

so K has d points.

(b) The Euler characteristic of $\mathbf{C}P^1 - K$ is $2 - d$, so that of its preimage is $2d - d^2$. The preimage of d small disks around the points of K is d. It follows that $\chi(V_d) = d + 2d - d^2 = 3d - d^2$.

2. **Complete bipartite graph question.** 20 POINTS A *bipartite graph* is one in which the vertices fall into two disjoint sets, say red and blue vertices, and each edge connects a red vertex to a blue one. It is *complete* if there is a unique edge connecting each red vertex to each blue one.

   Let $K_{m,n}$ denote the complete bipartite graph with m red vertices and n blue ones. Hence it has mn edges. For example, $K_{3,3}$ is the houses and utilities graph, which is known to be nonplanar.

   Show that if $K_{m,n}$ can be embedded in a closed oriented surface of genus g, then

$$g \geq \frac{(m-2)(n-2)}{4}.$$

Solution: If $K_{m,n}$ is embedded in such a surface, we get a polyhedron with $V = m + n$ vertices, $E = mn$ edges and F faces. If we add the number of edges on each face, we get $2mn$ since each edge is shared by two faces two faces. Each face must have at least four edges, so $2mn \geq 4F$ and $F \leq mn/2$. Thus the Euler characteristic of the surface is

$$\begin{aligned} 2 - 2g = V - E + F &= m + n - mn + F \\ &\leq m + n - mn + mn/2 = m + n - mn/2 \\ 2 - m + n + mn/2 &\leq 2g \\ g &\geq \frac{2 - m + n + mn/2}{2} = \frac{(m-2)(n-2)}{4} \end{aligned}$$

3. **Five lemma question.** (20 POINTS) The 5-lemma says that given a commutative diagram of abelian groups with exact rows,

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E', \end{array}$$

   if α, β, δ and ϵ are isomorphisms, then so is γ. Show by counterexample that the triviality of α, β, δ and ϵ does *not* imply the triviality of γ.

Solution: Let p be a prime. One counterexample is

$$\begin{array}{ccccccccc} 0 & \xrightarrow{i} & & \xrightarrow{j} & \mathbf{Z}/p^2 & \xrightarrow{k} & & \xrightarrow{\ell} & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ 0 & \xrightarrow{i'} & & \xrightarrow{j'} & \mathbf{Z}/p^2 & \xrightarrow{k'} & & \xrightarrow{\ell'} & 0. \end{array}$$

Another one is

$$\begin{array}{ccccccccc} 0 & \xrightarrow{i} & \mathbf{Z} & \xrightarrow{j} & \mathbf{Z} & \xrightarrow{k} & 0 & \xrightarrow{\ell} & 0 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ 0 & \xrightarrow{i'} & 0 & \xrightarrow{j'} & \mathbf{Z}/p^2 & \xrightarrow{k'} & \mathbf{Z} & \xrightarrow{\ell'} & 0. \end{array}$$

4. **Brouwer Fixed Point question.** (20 POINTS) Prove the 2-dimensional case of the Brouwer Fixed Point Theorem, i.e., that any continuous map of the 2-dimensional disk D^2 to itself has a fixed point. You may assume $\pi_1 S^1 = \mathbf{Z}$.

> **Solution:** See page 32 of Hatcher.

5. **James reduced product question.** (20 POINTS) Let the 2-sphere S^2 have base point x_0. Define an equivalence relation on the n-fold Cartesian product $(S^2)^{\times n}$ by saying that when a coordinate in a point

$$(x_1, x_2, \dots x_n) \in (S^2)^{\times n}$$

is the base point x_0, it may be transposed with either the coordinate on its left or the one its right. For example the points

$$(x_0, x_1, x_2),\ (x_1, x_0, x_2)\ \text{and}\ (x_1, x_2, x_0)\ \in (S^2)^{\times 3}$$

are all equivalent for arbitrary points $x_1, x_2 \in S^2$. The space $J_n S^2$ of equivalence classes in $(S^2)^{\times n}$ is called the *nth James reduced product of* S^2, having first been studied by Ioan James in the 1950s. One could make a similar definition with S^2 replaced by any pointed space. Thus there is a surjective map $f_n : (S^2)^{\times n} \to J_n S^2$.

We know that $(S^2)^n$ has a CW-structure with $\binom{n}{k}$ cells in dimension $2k$ for $0 \le k \le n$. We also know that as a ring under cup product,

$$H^*((S^2)^{\times n}; \mathbf{Z}) \cong \mathbf{Z}[y_i : 1 \le i \le n]/(y_i^2),$$

with $y_i \in H^2$ being the generator associated with the ith factor of the Cartesian product.

It can be shown that $J_n S^2$ has a CW-structure with a single cell in every even dimension up to $2n$. For $1 \le k \le n$, the group $H^{2k}(J_n S^2; \mathbf{Z}) \cong \mathbf{Z}$ has a generator u_k whose image under f_n^* is the sum of all square free k-fold products of the y_is.

Use this information to determine the cup product structure of $H^*(J_n S^2; \mathbf{Z})$. Give a formula for $u_k u_\ell$ as a multiple of $u_{k+\ell}$ for $k + \ell \le n$. In particular f_n induces a monomorphism in cohomology.

HINT: Try doing this first for small values of n such as 2, 3 and 4.

> **Solution:** The binomial theorem implies that in $H^*((S^2)^{\times n}; \mathbf{Z})$,
>
> $$\left(\sum_{1 \le i \le n} y_i \right)^k = k! f_n^* u_k.$$
>
> Since the sum on the left is $f_n^* u_1$, this implies that
>
> $$u_1^k = k! u_k.$$

Formally we can write

$$u_k = \frac{u_1^k}{k!}, \qquad u_\ell = \frac{u_1^\ell}{\ell!} \qquad \text{and} \qquad u_{k+\ell} = \frac{u_1^{k+\ell}}{(k+\ell)!},$$

which implies that

$$u_k u_\ell = \frac{(k+\ell)!}{k!\ell!} u_{k+\ell} = \binom{k+\ell}{k} u_{k+\ell}.$$