Math 430 Take-Home Midterm Due November 10

1. Let A be a Dedekind domain in a field K, and let L and L' be finite separable extensions of K, with B and B' the integral closures of A in L and L' respectively. Suppose there is a prime $\mathfrak{p}$ of A such that $\mathfrak{p}B = \mathfrak{q}^{[L:K]}$ (that is, $\mathfrak{p}$ ramifies completely in B) and $\mathfrak{p}$ does not ramify in B'. Show that $L \cap L' = K$. [Hint: How would $\mathfrak{p}$ factor in the integral closure of A in $L \cap L'$?]

2. Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$, let $\phi : R \longrightarrow R/\mathfrak{m}^2$ and suppose that $x_1, \ldots, x_n \in \mathfrak{m}$ have the property that $\phi(x_1), \ldots, \phi(x_n)$ generate $\mathfrak{m}/\mathfrak{m}^2$ as an R-module. Show that $x_1, \ldots, x_n$ generate $\mathfrak{m}$ as an R-module [Hint: Let N be the module generated by $x_1, \ldots, x_n$. Show that $M/N = 0$]

3. 1. Let L be a degree n field extension of $\mathbb{Q}$. Let $B \subset L$ be a ring that is integral over $\mathbb{Z}$ and has field of fractions L. Let $\sigma_1, \ldots, \sigma_n$ be the n distinct embeddings $\sigma : L \longrightarrow \mathbb{C}$. Show that for any basis $w_1, \ldots, w_n$ for B as an A-module, we have

$$\Delta(B/\mathbb{Z}) = \left(\det[\sigma_i(w_j)]\right)^2.$$

[Hint: Multiply $[\sigma_i(w_j)]$ by its transpose and use the fact (that you should prove, using results from class) that $\mathrm{T}_{L/K}(y) = \sigma_1(y) + \cdots + \sigma_n(y)$ for any $y \in L$.]

4. (30 points) Suppose that $d > 1$ is a square-free integer such that $d \equiv 1 \pmod 9$.

(a) Find a formula for the coefficients of the minimal monic polynomial f of $\theta_d = \frac{1+\sqrt[3]{d}+\sqrt[3]{d^2}}{3}$. Your answer should be a degree 3 polynomial. You may calculate f by looking at the characteristic polynomial for the multiplication-by-θ_d map on $\mathbb{Q}(\sqrt[3]{d})$ (or by any method that you choose). Feel free to use a matrix calculator or any other computation aid. Your formula should be quite simple. (It is very easy to see what the degree 2 term will be and the linear and constant terms have simple formulas in terms of d, though it may take you a while to find these formulas.)

(b) Show that θ_d is integral.

(c) Let $B = \mathbb{Z}[\sqrt[3]{d}, \theta_d]$. Find $|\Delta(B/\mathbb{Z})|$ and show that B is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt[3]{d})$. [Hint: There are not that many candidates for $|\Delta(B/\mathbb{Z})$ since it must be $|\Delta(\mathbb{Z}[\theta_d]/\mathbb{Z})|$ divided by a square.]

(d) Let $d = 10$. Show that $\mathbb{Z}[\theta_d]$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt[3]{d})$ by calculating $\Delta(\mathbb{Z}[\theta_d]/\mathbb{Z})$. (You may use any kind of calculation help you like for the discriminant.)

(e) Let $d = 19$. Show that $\mathbb{Z}[\theta_d]$ is *not* the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt[3]{d})$ by calculating the $\Delta(\mathbb{Z}[\theta_d]/\mathbb{Z})$. (You may use any kind of calculation help you like for the discriminant.)

5. Exercise 4 from Janusz, page 58. Restrict to the case where m is a prime number. (This simplifies somes versions of the proof.)

6. Exercise 7 from Janusz, page 58.