Math 430 Final Exam, Due December 18

1. Find the class group of $\mathbb{Z}[\sqrt{-14}]$. Show all of your work.

2. Let $d \equiv 3 \pmod 4$ be a square-free positive integer and write $d = p_1 \cdots p_m$ for primes p_i. For each p_i, let $\mathfrak{q}_i$ denote the unique prime in $\mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ lying over p_i. Let $1 < m < d$ be a divisor of d, and let I be an integral ideal of $\mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ such that $|N(I)| = m$. Let $[I]$ denote the image of I in the class group of $\mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$.

(a) Show that $[I] \neq 1$ (i..e I is not principal). [Note that this is easy when $m \neq d/3$. It requires a special, though not especially difficult argument if $m = d/3$.]
(b) Show that $[I^2] = 1$ (i.e. $[I]$ has order 2 in the class group).
(c) Show that there is a unique integral ideal $J \neq I$ with $|N(J)|$ dividing d such that $[IJ] = 1$. [Hint: What should $|N(J)|$ be?]

3. Let d be a square-free integer with $d > 1$. Suppose that the ring of integers in $\mathbb{Q}(\sqrt{-d})$ is a principal ideal domain (i.e. the class group is trivial). Note that on an earlier homework, we showed we must then have $d \equiv 3 \pmod 4$, so this ring of integers is always $\mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$. We also showed (both in the previous problem and earlier) that if d is not a prime, then the class group must be even, so we may assume that d is a prime number if we like.

Show that

(a) If $d > 8$, then we must have $d \equiv 3 \pmod 8$. [Hint: Consider the primes lying over 2 and note that our ring is of the form $\mathbb{Z}[\omega]$ where $\omega^2 - \omega + (d+1)/4 = 0$.]
(b) If $d > 12$, then we must have $d \equiv 1 \pmod 3$. [Hint: Consider the primes lying over 3.]
(c) If $d > 20$, then we must have $d \equiv 3 \pmod 5$ or $d \equiv 2 \pmod 5$. [Hint: Consider the primes lying over 5.]
(d) Show that if $d > 20$, we must have $d \equiv 43 \pmod{120}$ or $d \equiv 67 \pmod{120}$. [Hint: Use the previous three parts plus the Chinese remainder theorem.]

4. Which of the following polynomials have roots in $\mathbb{Q}_3$? (Explain your answers). [You may use Hensel's lemma and the fact that $\mathbb{Z}_3$ is a DVR.]

(a) $x^2 + 2$;
(b) $x^3 + x + 1$;
(c) $x^2 + 2x - 2$.

5. For polynomials $f/g \in \mathbb{Q}(t)$, with $f, g \neq 0$, we define $v_t(f/g) = (\deg g) - (\deg f)$.

(a) Let $|h| = e^{-v_t(h)}$ for nonzero $h \in \mathbb{Q}(t)$ and let $|0| = 0$. Show that $|\cdot|$ is a valuation on $\mathbb{Q}(t)$ (i.e. is multiplicative and satisfies the triangle inequality).
(b) Let $\hat{K}$ denote the completion of $\mathbb{Q}(t)$ with respect to $|\cdot|$. Let U be the set of elements x of $\hat{K}$ with $|x| = 1$. Show that U is not compact. Explain your answer carefully.