Math 430 Tom Tucker
NOTES FROM CLASS 11/11

Recall a quick preview of what we are going to do.

We want to show that there is an element of small norm in I. To make the proof of the finiteness of the class number as clear as possible, we'll first give simple versions of it and then prove more quantitative versions later.

Theorem 19.1. *(Imprecise small element of fractional ideal) There exists a constant $C(L)$ depending only on L such that for any fractional ideal I of $\mathcal{O}_L$ there is an element $y \in I$*

$$\mathrm{N}_{L/K}(y) \leq C(L)\,\mathrm{N}_{L/K}(I).$$

Theorem 19.2. *Assume Theorem 19.1 above. For any fractional ideal I of $\mathcal{O}_L$, there is an ideal $J \subset \mathcal{O}_L$ in the same ideal class as I such that*

$$|\,\mathrm{N}_{L/\mathbb{Q}}(J)| \leq C(L).$$

Proof. By Theorem 19.1 above, there exists $a \in I^{-1}$ such that

$$|\,\mathrm{N}_{L/\mathbb{Q}}(a)| \leq |\,\mathrm{N}_{L/\mathbb{Q}}(I^{-1})|C(L).$$

Then $J = Ia \subseteq \mathcal{O}_L$ and

$$|\,\mathrm{N}_{L/\mathbb{Q}}(J)| \leq C(L).$$

□

We'll need Minkowski's theorem, which guarantees the existence of certain elements of a lattice.

Lemma 19.3. *Let $\mathcal{L}$ be a lattice in V ($\mathbb{R}^n$ with a volume form) and let U be a measurable subset of V such that the translates $U + \lambda$, where $\lambda \in \mathcal{L}$ are disjoint. Then* $\mathrm{Vol}(U) \leq \mathrm{Vol}(\mathcal{L})$.

Proof. Let $\mathcal{T}$ be a fundamental parallelepiped for some basis of $\mathcal{L}$. For each $\lambda \in \mathcal{L}$, let

$$U_\lambda = \mathcal{T} \cap (U - \lambda).$$

We then have

$$U = \bigcup_{\lambda \in \mathcal{L}} (U_\lambda + \lambda).$$

Since the volume form is translate invariant, we see that

$$\sum_{\lambda \in \mathcal{L}} \mathrm{Vol}(U_\lambda) = \sum_{\lambda \in \mathcal{L}} \mathrm{Vol}(U_\lambda + \lambda) = \mathrm{Vol}(U).$$

Since all the U_λ are disjoint and contained in $\mathcal{T}$, we see that

$$\mathrm{Vol}(\mathcal{L}) = \mathrm{Vol}(\mathcal{T}) \geq \mathrm{Vol}(\bigcup_{\lambda\in\mathcal{L}}(U_\lambda)) = \sum_{\lambda\in\mathcal{L}} \mathrm{Vol}(U_\lambda) = \mathrm{Vol}(U).$$

□

Theorem 19.4. *(Minkowsi) Let $\mathcal{L}$ be a full lattice in the volumed vector space V of dimension n and let U be a bounded, centrally symmetric, convex subset of V. If $\mathrm{Vol}(U) > 2^n \mathrm{Vol}(\mathcal{L})$, then U contains a nonzero element $\lambda \in \mathcal{L}$*

Proof. By the way, centrally symmetric means that for $x \in U$, we have $-x \in U$. Convex means that for $x, y \in U$ and $t \in [0,1]$, we have $tx + (1-t)y \in U$.

Now, let $W = \frac{1}{2}U$. Then $\mathrm{Vol}(W) = \frac{1}{2^n}\mathrm{Vol}(U)$, so $\mathrm{Vol}(W) > \mathrm{Vol}(\mathcal{L})$, so it follows from the Lemma, we just proved that not all of the translates $W+\lambda$ are disjoint. Taking $y \in (W+\lambda)\cap(W+\lambda')$, with $\lambda \neq \lambda'$, we can write $y = a+\lambda = b+\lambda'$, which gives us $a, b \in W$ with $(a-b) \in \mathcal{L}$ and $(a-b) \neq 0$. Since $a, b \in W = \frac{1}{2}U$, we can write $a = \frac{1}{2}x$ and $b = \frac{1}{2}y$ for $x, y \in U$. Since y is convex and centrally symmetric the element $a - b = \frac{1}{2}x - \frac{1}{2}y = \frac{1}{2}x + \frac{1}{2}(-y) \in U$ and we are done. □

We will want to apply this to a lattice $h(I)$ for I a fractional ideal of $\mathcal{O}_L$. The region U that we use should consist of elements of bounded norm. Recall though, that the most natural sort of region is something like a sphere $\sqrt{x_1^2 + \cdots + x_n^2} \leq M$ and we are going to be interested in something like the product $x_1 \cdots x_n$, so we will need something relating these two. Also, we have messed around a bit at the complex places, to we'll have to tinker with that a bit. Let's label our coordinate system for V in the following way. We call the first r-coordinates corresponding to the real embeddings $x_1, \ldots, x_r$. The remaining $2s$ coordinates we label as $y_1, z_1, \ldots, y_s, z_s$.

Let

$$X_t = \{x_1, \ldots, x_r, y_1, z_1, \ldots, y_s, z_s \mid \sum_{i=1}^{r} |x_i| + \sum_{j=1}^{s} 2\sqrt{y_j^2 + z_j^2} \leq t\}$$

from now on. It is easy to see that X_t is convex, bounded, and centrally symmetric, so we will be able to apply Minkowski's theorem to it.