Q: If $F_n \Rightarrow F$, show $\exists$ rvs $\{X_n\}, X$ on $(\Omega, \Sigma, \mathbb{P})$ st $X_n \to X$ a.s.

$X_n(y) = \inf\{x : F_n(x) \geq y\}$ (see Ch 8 in Le Gall)

- $X(y)$ is increasing since if $y_1 < y_2$ $\{x : F(x) \geq y_2\} \subset \{x : F(x) \geq y_1\}$

$\Rightarrow X(y_1) \leq X(y_2)$.

- $X(y)$ is left continuous: let $y \uparrow y_0$ Suppose $X(y_0) > \lim_{y \uparrow y_0} X(y)$. Then

By right continuity of F, $F(X(y_0)) \geq y_0$

$\lim_{y \uparrow y_0} F(X(y)) \geq \lim_{y \uparrow y_0} y = y_0$ So for any $\lim_{y \uparrow y_0} X(y) < t < X(y_0)$

$F(t) \geq y_0 \Rightarrow X(y) \leq t$. This is a contradiction.

So the points of discontinuity of $X(y)$ for $y \in (0,1)$ must be countable.

This is because for any y, $\exists x$ st $F(x) \geq y$. So $X_n(y) \leq x < \infty$.

If there were an uncountable # of points of discontinuity, $X_n(y)$ would be $+\infty$ at some $y \in (0,1)$.

Let $\{x_j\}_{j=1}^{\infty}$ be the points of discontinuity of F.

Then $F(x_j^-) < F(x_j^+)$

So when F jumps, and $F(x_j^-) \leq y \leq F(x_j^+)$

$X(y) = x_j$, a constant.

Conversely, when F is flat, X jumps: this is because if $F(x) = c$ on $[a,b)$ Then $X(c^-) = a$ and $X(c^+) = b$

Of course $[a,b)$ is the largest interval st $F(x) = c$.

Let y_0 be a point of continuity for $X(y)$. Then for any $t > X(y_0)$ that is a point of continuity of F, $F_n(t) \to F(t)$.

Moreover F is not constant on an interval containing $X(y_0)$.

$$F(t) > F(X(y_0)) \Rightarrow F_n(t) > F(X(y_0)) \geq y_0$$

$$\Rightarrow X_n(y_0) \leq t . \Rightarrow \overline{\lim}\, X_n(y_0) < t$$

If $F(t') < y_0 \Rightarrow t' \notin \{x : F(x) \geq y_0\}$ which is closed.

$\Rightarrow X(y_0) > t'$.

So take any $t' < X(y_0) \Rightarrow F(t') < y_0 \Rightarrow \lim F_n(t') < y_0$

$$\Rightarrow X_n(y_0) > t' . \Rightarrow \underline{\lim}\, X_n(y_0) > t'$$

Let t' and t approach y_0 through points of continuity of F.
Since the discontinuity points of F are countable, this can be done.

Thus $X_n(y) \to X(y)$ on all pts of continuity of y.

$\Rightarrow X_n(y) \to X(y)$ a.s.