Math 403: Introduction to Probability

Midterm 1

March, 2020

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Instructions:

1. Read the notes below:
   - **Using any notes, books, online resources, or contacting any other people during this exam is prohibited.**
2. Read the following Academic Honesty Statement and sign:

   I affirm that I will not use any unauthorized resourced, or give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:______________________________

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

1. (10 points) Let $\Omega = \{0,1\}^{\mathbb{Z}}$ be the bi-infinite space of sequences of zeroes and ones, endowed with the cylindrical sigma-algebra. An element $\omega \in \Omega$ thus has the form

$$\omega = \ldots \omega_{-n} \ldots \omega_{-1}\omega_0\omega_1 \ldots \omega_n \ldots,$$

where $\omega_n \in \{0,1\}$ for all n. Prove that the cylindrical sigma-algebra is generated by the cylinders of the form $C_n^\alpha := \{\omega | \omega_n = \alpha\}$ (here $n \in \mathbb{Z}, \alpha \in \{0,1\}$.)

Remark: The cylindrical sigma algebra on M^∞ is the algebra generated by the sets $B_1 \times B_2 \times \cdots \times B_k \times \ldots$ such that only finitely many of the B's are not the full space M.

2. (10 points) Let $\{X_n : n \geq 1\}$ be independent and exponentially distributed with parameter 1. Compute

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{\ln n} = 1\right).$$

Hint available at the back of the exam. Solving without the hint will get you a bonus of 3 points.

Remark: The exponential distribution with rate λ has density $\lambda e^{-\lambda x}$ and is supported on $[0, \infty)$.

Did you use the hint? Yes No

3. (10 points)

Let $X_1, X_2, \dots$ be a sequence of positive i.i.d. random variables. Prove that if $P(X_i > n) < \frac{1}{n^2}$ for all positive integers n, then

$$\frac{\max\{X_1, \dots, X_n\}}{n} \to 0 \text{ in probability.}$$

4. (10 points) Let $X_1, X_2, \dots$ be independent random variables such that

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2n\log n}, \quad P(X_n = 0) = 1 - \frac{1}{n\log n}.$$

Let $S_n = X_2 + \cdots + X_{n+1}$. Prove that

$$\frac{S_n}{n} \to 0 \text{ in probability.}$$

Show that for every $\varepsilon > 0$ we have

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{\ln n} > 1 + \varepsilon\right) = 0$$

and

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{\ln n} > 1 - \varepsilon\right) = 1$$