MTH 248 - Homework 8

Question 1. Recall in general the Hungarian algorithm is a method to find a maximum weight matching in a weighted bipartite graph (and a dual minimum cost vertex cover). It is efficient, taking roughly $O(nm)$ steps, where n is the number of vertices, m is the number of edges, although it is a bit complicated. (In $K_{n,n}$ there are $n!$ perfect matchings so a direct weight comparison of these perfect matchings is in general too inefficient to be effective once $n > 20$ and thus the more complicated Hungarian algorithm is a necessity.) In this exercise, you are asked to work through the problem with the Hungarian algorithm even though the number of vertices is small enough that other methods will work. Consider the graph $K_{5,5}$ with weights given by the weight matrix

$$W = \begin{bmatrix} 7 & 8 & 9 & 8 & 7 \\ 8 & 7 & 6 & 7 & 6 \\ 9 & 6 & 5 & 4 & 6 \\ 8 & 5 & 7 & 6 & 4 \\ 7 & 6 & 5 & 5 & 5 \end{bmatrix}$$

Recall that w_{ij} gives the weight of the x_i, y_j edge in $K_{5,5}$, where $X = \{x_1, \cdots, x_5\}$ and $Y = \{y_1, \cdots, y_5\}$ are the partite sets. The first step in the algorithm is to choose a vertex cover $(u, v) = ((u_1, \cdots, u_5), (v_1, \cdots, v_5))$, where u_i is a label for x_i and v_j for y_j, and $u_i + v_j \geq w_{ij}$.

(a) Using an initial vertex cover with $(v_1, \cdots, v_5) = (0, \cdots, 0)$, write down the excess matrix with entries $\epsilon_{ij} = u_i + v_j - w_{ij}$, draw the equality subgraph $G_{u,v}$, and find a maximum size matching in $G_{u,v}$.

(b) Adjust the vertex cover as determined by the Hungarian algorithm and repeat the process in part (a) with your new cover.

(c) Continue in this way until a maximum weight matching and minimum cost vertex cover are found. Explicitly show the matching and vertex cover found with edge weights, (u, v) labels, total weight of the matching $w(M)$, and cost of the vertex cover $c(u, v)$. Explain how you know your matching and cover are optimal.

Question 2. Let T be a tree with $n(T) > 1$. Find $\kappa(T), \kappa'(T), \delta(T)$ and justify your answers.

Question 3. Find the smallest 3-regular simple graph with connectivity 1.

Question 4. Let G be a simple, connected graph with $n(G) > 2$. Form G' from G by adding an edge x, y whenever $d_G(x, y) = 2$ (recall d_G is distance in G). Show that G' is 2-connected.

Question 5. Recall an x, y-cut in a graph G is a set $S \subseteq V(G)$ such that $G - S$ has no x, y-path.

(a) For any pair of non-adjacent vertices x, y in a graph G, we define the *local connectivity* of x, y, denoted $\kappa_G(x, y)$, as the minimum size of an x, y-cut in G.

Suppose G is simple and not complete. Explain why

$$\kappa(G) = \min \kappa(x, y),$$

where the min is taken over all pairs of non-adjacent vertices in G.

(b) For any pair of distinct vertices x, y in a graph G, we define the *local edge-connectivity* of x, y, denoted $\kappa'_G(x, y)$, as the minimum number of edges that need to be removed from G to disconnect x and y. Explain why

$$\kappa'(G) = \min_{x,y \in V(G)} \kappa'(x, y),$$

where the min is taken over all pairs of distinct vertices in G.

Question 6. The *join* of two graphs G_1 and G_2, denoted $G_1 \vee G_2$, is the graph obtained from the disjoint union of G_1 and G_2 by adding a single edge between every vertex of G_1 and every vertex of G_2. For example, if P and Q are paths of length 1, then $P \vee Q = K_4$.

(a) Show that if $e(H_1) \geq 1$ (and H_2 is non-empty), then $H_1 \vee H_2$ is not bipartite (Hint: is there an odd cycle?). Now suppose $G_1 \vee G_2$ is bipartite. What can you conclude about G_1, G_2, and $G_1 \vee G_2$?

(b) Let $J = K_r$ be the complete graph with r vertices and let G be a graph. Suppose S is a separating set (i.e. vertex cut) of $G \vee J$ such that $G \vee J - S$ has more than one vertex. Show that S contains all vertices from J. Conclude that $\kappa(G \vee K_r) = \kappa(G) + r$ for all $r \geq 1$.

Question 7. For any integers t, s, m with $0 < t \leq s \leq m$ let G be the graph constructed as follows. Take a disjoint union of two (m+1)-cliques K_{m+1} (we will call them the left clique L and right clique R respectively). Let $X = \{x_1, \cdots, x_s\}$ be s distinct vertices in left clique L and let $Y = \{y_1, \cdots, y_t\}$ be t distinct vertices in R. Let $f : X \to Y$ be any surjective/onto function (which is possible as $s \geq t > 0$.) Finally form G from $L \cup R$ by adding a single edge between $x_j \in X$ and $f(x_j) \in Y$ for each $j = 1, \cdots, s$.

(a) Show that G is a simple, connected graph which is not complete, and compute its minimum degree $\delta(G)$.

(b) Recall that $[V(L), V(R)] \subseteq E(G)$ is the set of edges in G that have one endpoint in $V(L)$ and the other endpoint in $V(R)$. Show that if S is a separating set of G, then $G - S$ contains no edges in $[V(L), V(R)]$ and conclude that $\kappa(G) = t$.

(c) Find $\kappa'(G)$.