MTH 248 - Homework 3

Textbook problems listed are from the text by West.

Question 1. Textbook 1.3.1.

Question 2. Textbook 1.3.2.

Question 3. Textbook 1.3.3. Note: recall for finite sets A, B that $|A \cup B| = |A| + |B| - |A \cap B|$. See the proof of proposition 1.3.15.

Question 4. Textbook 1.3.17.

Question 5. Textbook 1.3.40.

Question 6. Let G be a nonbipartite triangle-free simple graph with n vertices and $\delta(G) = k$. Let l be the minimum length of an odd cycle in G.

(a) Let C be a cycle of length l in G. Prove that every vertex not in $V(C)$ has at most two neighbors in $V(C)$.

(b) Count the edges joining $V(C)$ and $V(G) - V(C)$ in two different ways to prove $n \geq \frac{kl}{2}$. Hint: Count these edges by visiting each vertex of C and using the minimum degree. Then count these edges using the result in part (a).

Question 7.

(a) Prove that if all vertices of a graph have even degree then there is no cut-edge.

(b) Suppose that in some graph G the vertices $v_1, \cdots, v_J$ all have degree $J-2$. Now add vertices v_{J+1} and v_{J+2}, both with exactly one edge to v_i for every $i = 1, \cdots, J$, but adding no other edges (note: there is no v_{J+1}, v_{J+2} edge). For the resulting graph, explain why $v_1, \cdots, v_{J+1}, v_{J+2}$ all have degree J.

(c) For each $k \geq 1$, construct a $2k+1$-regular simple graph having a cut edge. Hint: try a recursive construction. Part (b) could help.

Question 8. Recall a graph is k-partite if its vertices can be partitioned into k independent sets, some of which can be empty.

(a) For each $k \geq 1$ and each loopless graph G, prove that G has a k-partite subgraph H with the same vertex set as G and with $e(H) \geq (1 - \frac{1}{k})e(G)$.
Hint: For $k \geq 2$ adapt the proof of the corresponding bipartite result done in class. Note that it will not matter if some of the sets in the initial k-partition $V = V_1 \cup \cdots \cup V_k$ that you use are empty.

(b) Find a bipartite subgraph of K_5 with the maximum number of edges and explain why your answer is indeed the maximum. What fraction of the edges of K_5 does your example use?