**1.1.4** $G \cong H$ (simple)

$\exists \; f : V(G) \to V(H)$ a bijection satisfying

$uv \in E(G)$ iff $f(u)f(v) \in E(H)$

The same map provides a bijection from $\overline{G}$ to $\overline{H}$

$uv \in E(\overline{G}) \Leftrightarrow uv \notin E(G) \Leftrightarrow f(u)f(v) \notin E(H) \Leftrightarrow f(u)f(v) \in E(\overline{H})$

(We only study isomorphism in the context of simple graphs here).

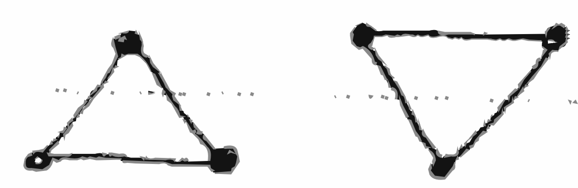
# HW 1 Answer Key

(Selected Solns)

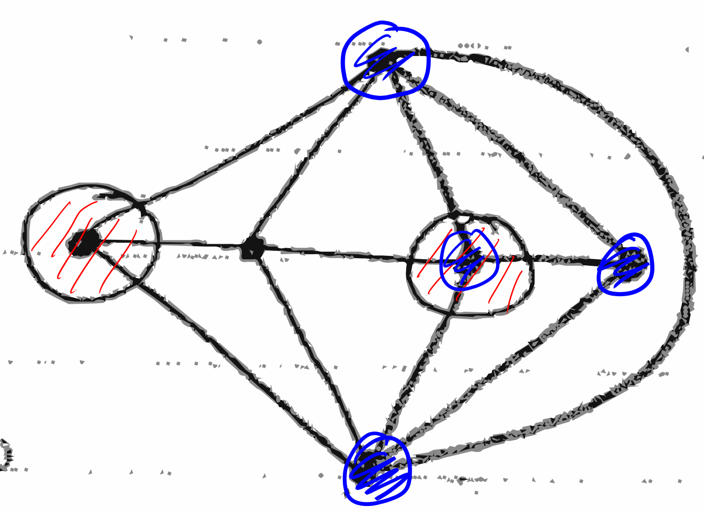
$\boxed{1.1.1}$ Only $K_{1,1}$ (or $K_{1,0}$, optionally) ~~are~~ is complete:

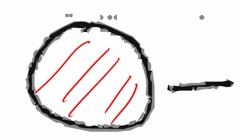
$K_{1,1}$:   $K_{1,0}$: 

$\boxed{1.1.5}$ False:



$\boxed{1.1.11}$



 — A maximal independent set.

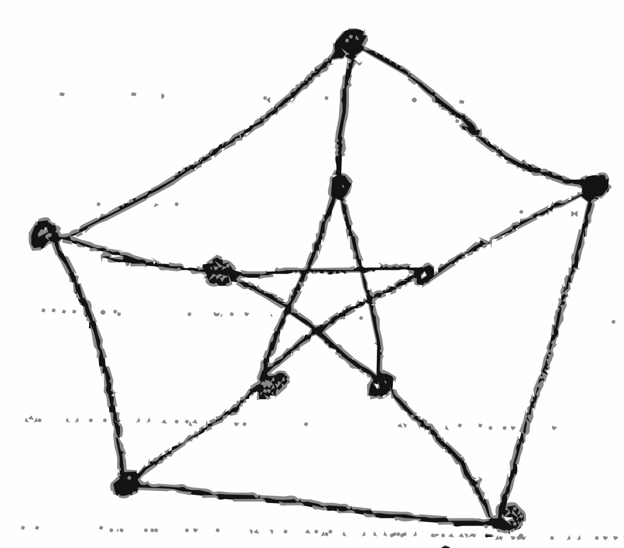
Cannot contain top/bottom, since adjacent to all. (indep set size = 1)

Can contain at most two middle vertices.

Maximal clique indicated in blue.

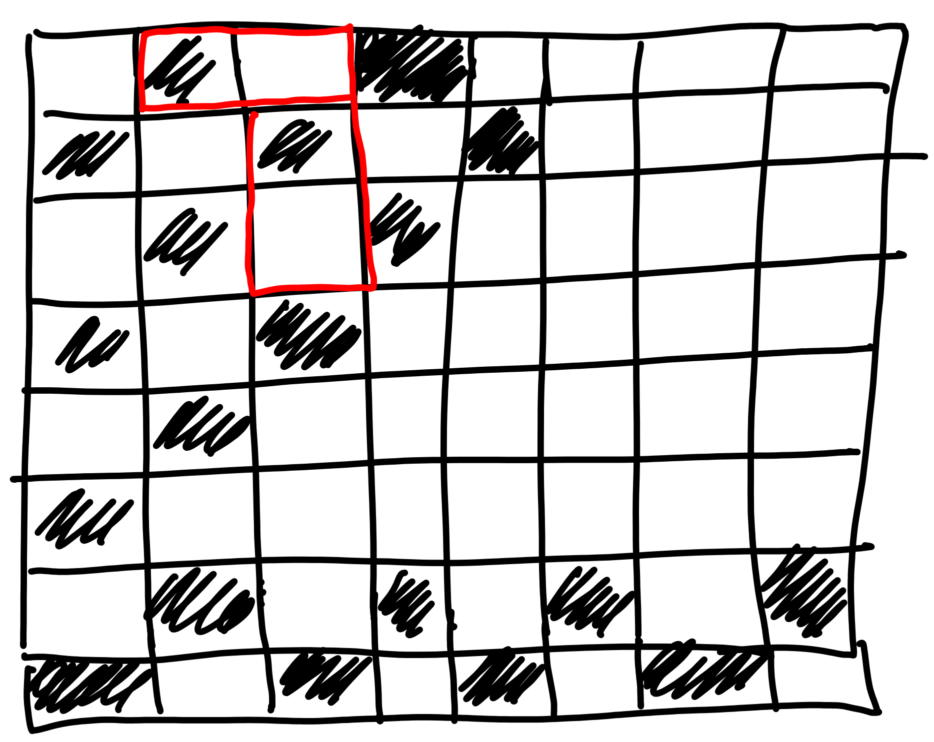
Easy to check no size 5 cliques: There are 6 size 5 sets to check.

$\boxed{1.1.12}$ The ~~pe~~ Peterson graph is not bipartite (it has a ~~C5~~ cycle of size 5 in it.)



No indep. set of size 5 – it would require either 3 in the inside cycle, or 3 in the outside cycle.

1.1.14 Prove that removing opposite corners of an 8x8 checkerboard leaves a subboard that cannot be partitioned into 1 by 2 and 2 by 1 rectangles.



let $X \sim$ black tiles

$Y \sim$ white tiles

Domino tiling = bipartite graph with $X$ $Y$ as bipartitions.

= bijection from $X$ to $Y$.

Remove two opposite corners, two white squares removed $\Rightarrow |Y'| = |Y| - 2$

There cannot exist a bijection from $X$ to $Y$.

In general a degree one regular bi-partite graph will have $|X| = |Y|$ for any bipartition.

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1.1.15 A : paths

B = cycles

C = complete graphs

D = bipartite graphs.

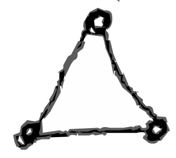
A and B - No cycles are also paths.

A and C - Only $K_1$ and $K_2$ are paths.

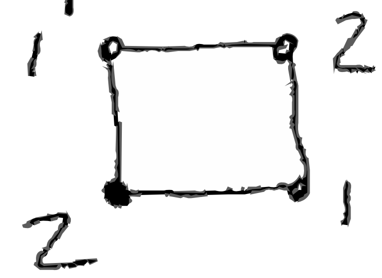
A and D - All paths are bipartite. $A \subset D$

If $P = \{a_1 \cdots a_n\}$ is a path. $X = \{a_i : i \text{ odd}\}$ $Y = \{a_i : i \text{ is even}\}$ is a bipartion

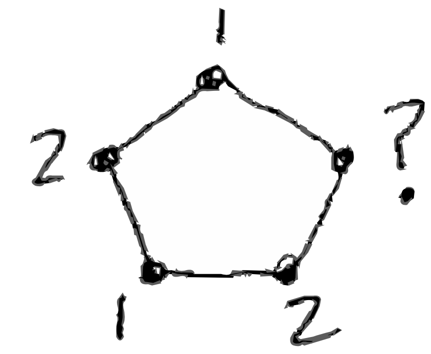
B and C – Only $K_3$ is a cycle: (for $K_r$, $r > 3$, $\deg(v) = r-1 \neq 2$) so it couldn't be a cycle

$K_3$: 

B and D – Cycles with an even number of vertices are bipartite.




vs

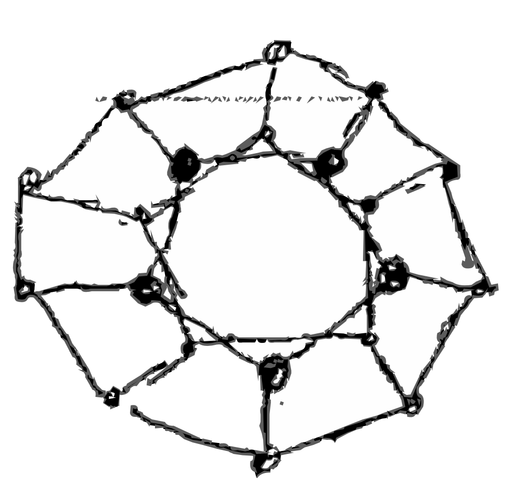



$C = c_1 \cdots c_k$ a cycle. Let $X = \{c_i : i \text{ is odd}\}$
$Y = \{c_i : i \text{ is even}\}$

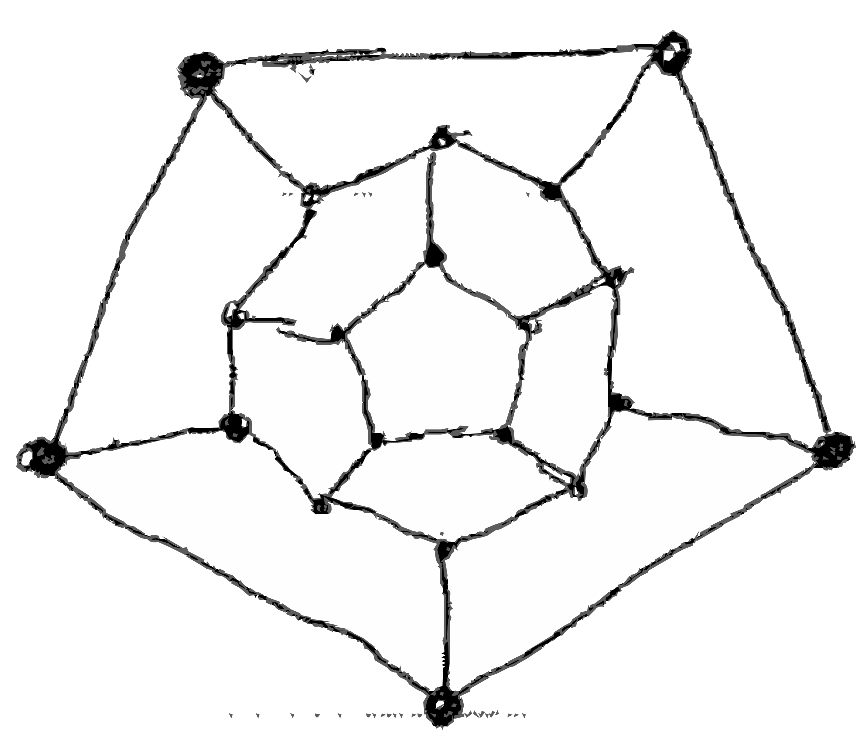
By Königs theorem, odd cycles cannot be bipartite.
↳ can also argue directly. Note that $C_2$ is not a cycle; the textbook requires cycles to have an equal # of edges & vertices.

C and D – Only $K_2$ is bipartite. (or $K_1$) for $K_r$, $r \geq 3$, easy to see it contains an odd cycle.

1.19 The first graph is bipartite, whereas the 2nd and 3rd are not. The 2nd and 3rd are isomorphic.
Hint:

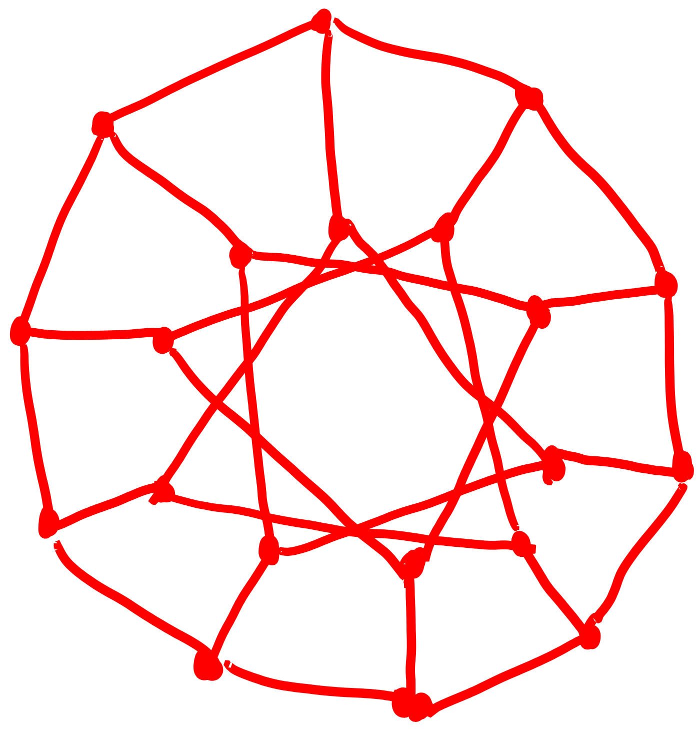


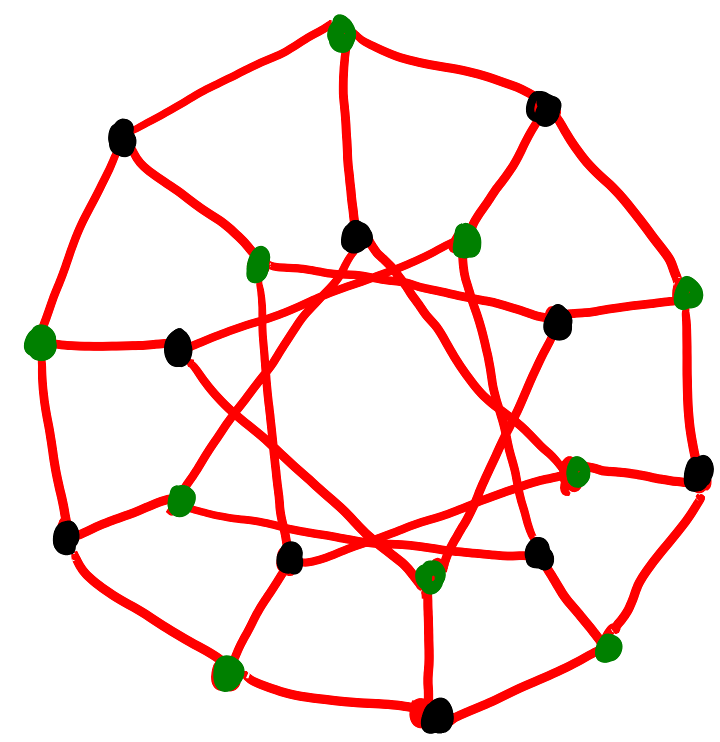
stretch →
inner pentagon



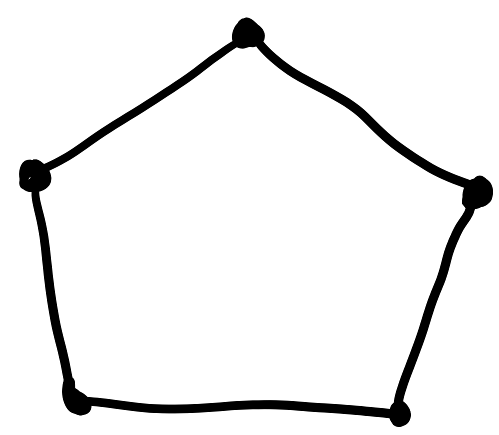
1.1.19

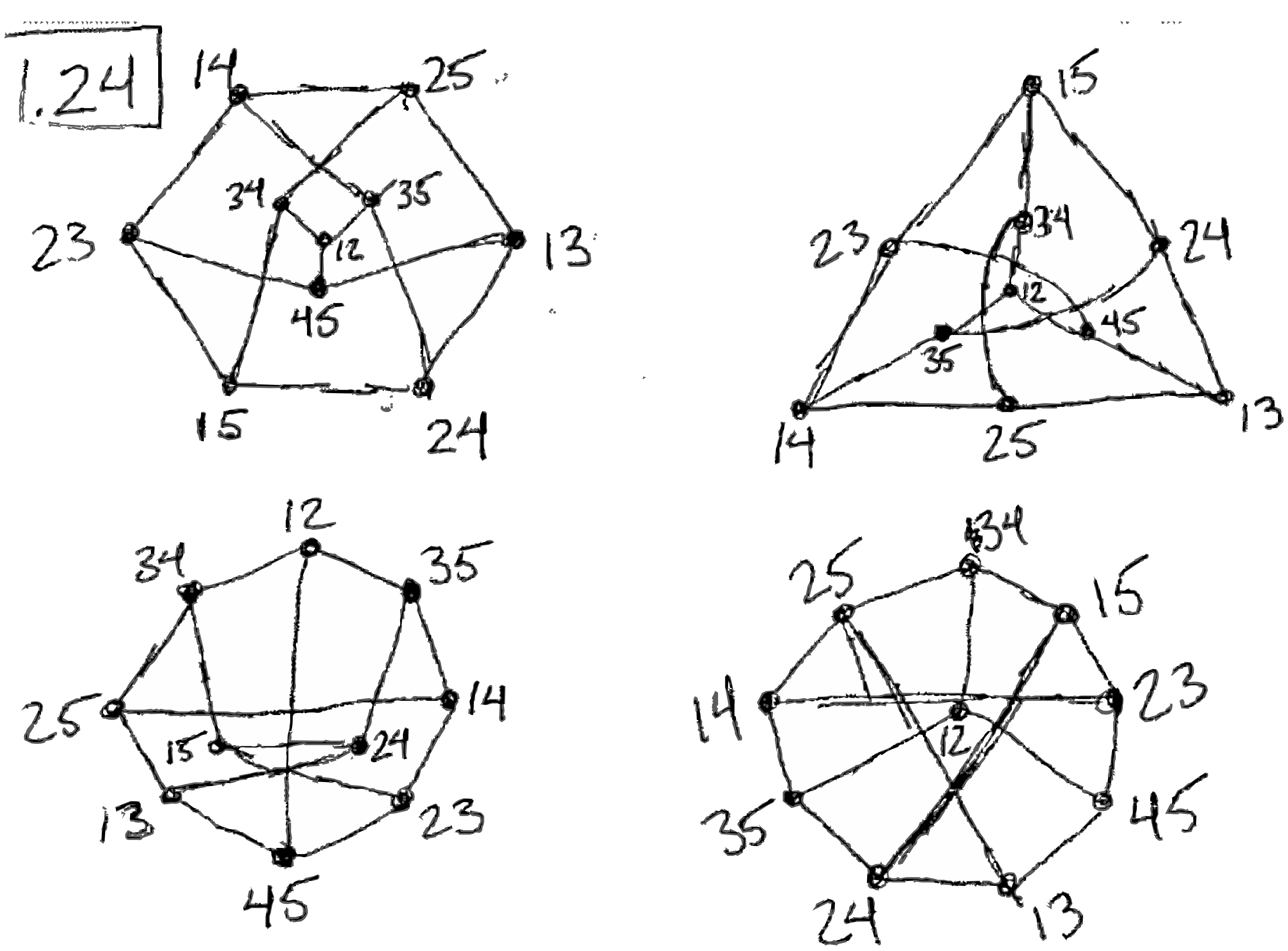
This is a bipartition.





In 3rd, you can take the inner pentagon and expand it out in the outer pentagon in the 2nd.

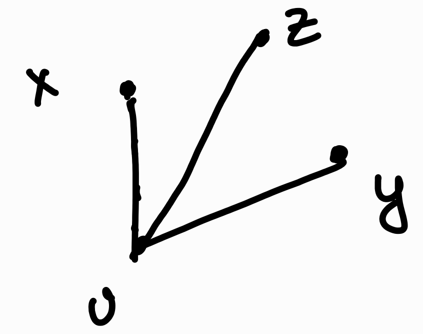




## 1.1.29

Draw the friend graph, vertices 1 ... 6.

Recall $\delta(G)$, $\Delta(G)$, minimal and maximal degrees. Let $v$ be st $d(v) = \Delta \geq 3$.



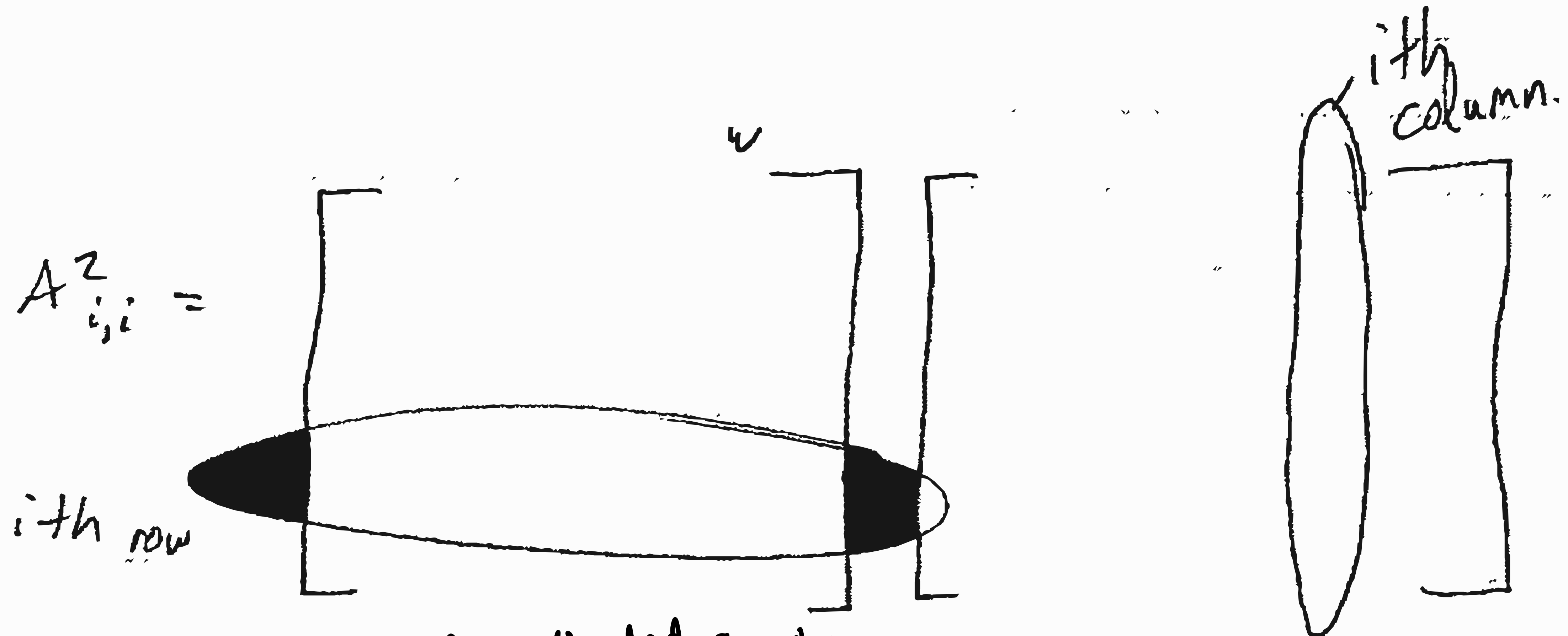

If $\exists$ an edge in $\{xyz\}$ then we have a 3 cycle, if not we have an indep set.

Since
$\delta(G) = 5 - \Delta(\bar{G})$, and so if $\delta(G) \leq 2 \Rightarrow \Delta(\bar{G}) \geq 3$ and the same argument applies.

We cannot simultaneously have $\delta(G) > 2$ $\Delta(G) < 3$

Q: What would your generalization of this be?

## 1.1.30




Adjacent matrices for undirected graphs are symmetric, so this is the ith row dotted with itself.

$(A^2)_{ij} = \sum_k A_{ik} A_{kj}$ (the terms of the sum are nonzero iff $A_{ik} A_{kj} \neq 0$)

In fact $A_{ik} \cdot A_{kj} =$ # of possible 2 step paths from $i$ to $j$ passing through $k$.

So summing over k

$(A^2)_{ij}$ = # of 2 step paths from i to j

indicator function

$M_{ij} = \mathbb{1}_{\{\text{an edge } j \text{ is incident on vertex } i.\}}$

$$(MM^T)_{ij} = \sum_k M_{ik} M^T_{kj} = \sum_k M_{ik} M_{jk}$$

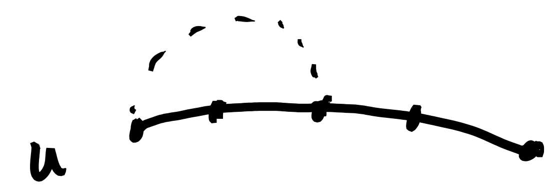
$M_{ik} M_{jk} = \mathbb{1}_{\{\text{vertices } i \text{ and } j \text{ have edge } k \text{ incident}\}}$

$\Rightarrow (MM^T)_{ij}$ = # of edges between vertex i and j.

G is given to be simple.

8 Take a maximal path in G

a)

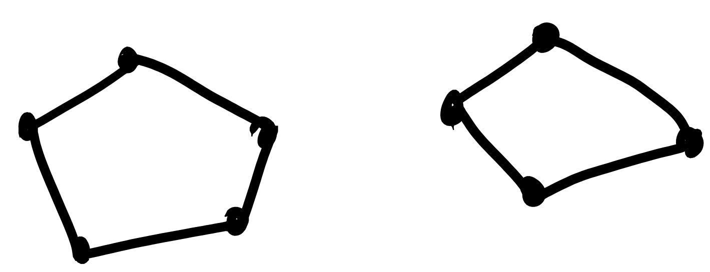



Its first vertex must have a neighbor v. v cannot be (since it has degree 2) in $P^c$; if it is we can extend P. So v must be on P

v cannot be a middle vertex since they all have deg 2.

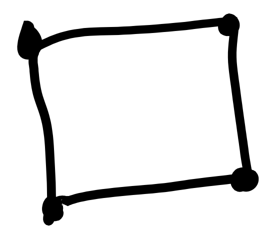
So v must be the final vertex. ∎

b)



2 cycles.

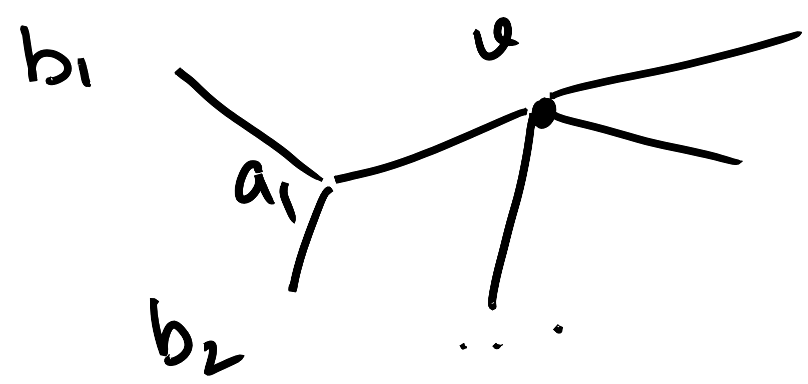
Every connected component must be a cycle.



$\exists$ a cycle of length 4 $\Rightarrow k \geq 2$.

a) loops are cycles of length 1
multiple edges are " of length 2

b) So pick any vertex $a_1 \in G$. It must have $k$ neighbours $\{b_1, \dots, b_k\}$




Now pick $b_1$; it must have $k$ neighbours. These cannot be in $\{b_2, \dots, b_k\}$ since if $b_1$ did have such a neighbor $b_j$ we would have $a_1 b_1 b_j$ forming a 3 cycle.

Thus $\exists$ $k-1$ new points $\{a_2, \dots a_{k-1}\}$ that are neighbours of $b_1$. So $G$ has at least
$|\{a_1, a_2, \quad, a_k, b_1 \dots b_k\}| = 2k$ vertices.

c) Pick $v$ as before, and let the bipartitions be
$X = \{a_1 a_2 \dots, a_k\}$ $\{b_1, \dots, b_k\} = Y$

edges between $Y$ forms a 3cycle as before $(a_1 b_i b_j)$
edges " $X$ form a 3cycle $(b_1 a_i a_j)$

So we have that $G$ is bipartite. It remains to show that $G$ is a biclique $K_{k,k}$. Each $a_i$ has degree $k$, and they must be connected to the $k$ vertices in $Y$. ∎