Math 248: Graph Theory

Midterm

Thursday, March 28th, 2024

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 14 pages.
- No calculators, phones, electronic devices, or books are allowed during the exam.
- You may bring a double-sided letter-sized page of notes.
- Show all work and justify all answers unless otherwise instructed.
- Read the instructions for each problem carefully.
- **There are 6 problems. You have to do 5 problems out of 6. Clearly state which of the 6 you are choosing to do. I will only grade those 5.**

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

1. (15 points)

For each of the following statements, determine if it is true or false. Justify your answer with an example or an argument.

(a) Two graphs G and H are isomorphic iff their adjacency matrices are identical.

(Isomorphic)

$f : \{1,2,3\} \to \{1,3,2\}$ obvious

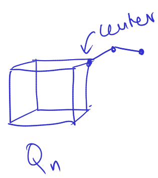
$f : \{12, 23\} \to \{13, 32\}$ bijection.

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

FALSE.

(b) For any integer $n \geq 2$, there exists a simple graph with radius n and diameter $n + 2$.

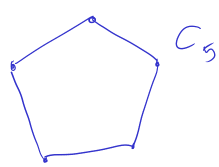
Certainly $\text{diam}(G) \leq 2\,\text{rad}(G)$. $n+2 \leq 2n \Rightarrow n \geq 2$ is necessary.



diam $= n+2$

True.

(c) Every Eulerian graph has a cut-edge.



False

2. (15 points)

(a) Given a connected graph of n vertices, what is the minimal number of edges it can have? Justify.

# of components $\geq n - e(G)$

Add an edge one at a time, each edge can either reduce components by 1 or leave them the same. We start n components.

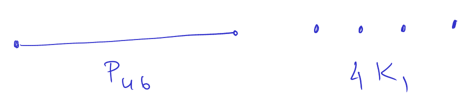
Connected $\Rightarrow$ 1 component $\Rightarrow$ $e(G) \geq n-1$

A tree realizes this.

(b) Given a graph with 50 vertices and degree sum 90, find the minimal number of components it must have, and find a graph that achieves the minimum. Prove your answer.

By previous # components $\geq n - e(G) = 50 - \underbrace{\frac{90}{2}}_{\text{degree sum}} = 5$

Realized by



simple

(c) Given a graph with 50 vertices and degree sum 90, find the maximum number of components it must have, and find a graph that achieves the maximum. No proof necessary.

Claim: Given two simple graphs G, H. Let r be the smallest # st $\frac{r(r-1)}{2} \geq e(G)+e(H)$

Then $r < n(G)+n(H)$.

$e(\cdot) \leq \frac{n(\cdot)(n(\cdot)-1)}{2} \Rightarrow e(G)+e(H) \leq \frac{n(G)(n(G)-1)}{2} + \frac{n(H)(n(H)-1)}{2}$

$< \frac{(n(G)+n(H))(n(G)+n(H)-1)}{2}$

(It follows from $a(a-1)+b(b-1) < (a+b)(a+b-1)$ if $a,b>0$.)

$\Leftrightarrow a^2+b^2-(a+b) < (a+b)^2-(a+b)$

This implies $A + (n(G)+n(H)-r)K_1$ has at least two components and in general more components than $G+H$, where $n(A)=r$.

Then by induction one can argue that $A + (n-r)K_1$ maximizes the # of component. For $n=50$, $e=45$ Choose $A=K_r$ $r=10$.

Which gives $K_{10} + (50-10)K_1$ which has 41 components

→ Since I forgot to add the word simple. You may also do 50 components by adding 45 loops.

3. (15 points)

(a) Show that an acyclic graph is a disjoint union of trees.

Let $G = A_1 + {}_2 + \cdots A_n$ where A_i are the components of G.

Each connected component A_i has $n(A_i) - 1$ edges $\Rightarrow$ its a tree.

(b) d is a sequence of length 100 of the form $(50, 50, \cdots, 49, 49, 49, 49)$. That is, there are 96 entries of degree 50, and 4 entries of degree 49. Does there exist a graph with this degree sequence? If it does, find it, if not, prove that it does not exist.

You could use Havel-Hakimi for this. But notice that its like taking a 49-regular graph on 100 vertices pairing up 96 of them and adding edges.

Take $K_{50} + K_{50}$ which is 49 regular. Pair up 96 of them and add an edge to give 96 vertices of degree 50 and 4 of degree 49.

4. (15 points)

1. On the vertex set $\{1, \ldots, n\}$, describe all possible Prüfer codes of tree with exactly two leaves. Use this to count the total number of trees with exactly two leaves on n vertices.

If the tree has exactly two leaves, all other vertices must have degree 2. Moreover, the leaves must be attached to different vertices.

There are n choices for the smallest leaf to attach to. There are $n-1$ for the next. Now we have a tree on $n-2$ vertices with exactly 2 leaves, since removing the 2 leaves reduces the degree of the vertices they are attached to by 1.

Arguing inductively $\underbrace{\underline{n}\ \underline{n-1}\ \underline{n-2}\ \ldots\ \underline{3}}_{n-2}$

and so the # of Prüfer codes is $n(n-1)\cdots 3 = \frac{n!}{2}$

You can also argue this by saying if you have a path on $\{1, \ldots, n\}$ corresponding to the permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$. The only automorphism is $(\sigma_n, \ldots, \sigma_1)$ (reversal).

2. Let G be a simple connected graph. Prove that $v \in V(G)$ is a cut-vertex iff no spanning tree of G has v as a leaf.

$\Rightarrow$ Suppose $\exists$ a tree T with v as a leaf. Then $T - v$ is connected. It has $n(G)-2$ edges and thus its a spanning tree for $G-v \Rightarrow G-v$ is connected $\Rightarrow v$ is not a cut vertex. (We have proved the contrapositive of $\Rightarrow$)

$\Leftarrow$ Suppose v is not a cut vertex. Then $G-v$ is connected. Take any spanning tree T' of $G-v$. Attach v to any vertex of T'. $T'+v$ is a spanning tree of G, and v is a leaf by construction. (Again proved the contrapositive)

Most of you could see how to do this. It was an exercise in proof writing.

5. (15 points)

(a) Using the matrix-tree theorem, count the number of trees on 4 vertices by computing a determinant

This is equivalent to counting the spanning trees of K_4.

$$D-A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$C_{11} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 4 & -4 & 0 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 4 & -4 & 0 \\ 0 & 4 & -4 \\ -1 & -1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & -4 \\ -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -1 & -2 & 1 \end{vmatrix} = 4^2$$

(b) Prove Cayley's formula (for general n) using the matrix-tree theorem.

$$C_{11} = \begin{vmatrix} n-1 & -1 & \cdots \\ -1 & n-1 & \cdots \\ & & \\ & & \end{vmatrix}_{n-1\times n-1} \overset{R_1 = R_1 - R_2}{=} \begin{vmatrix} n & -n & 0 & 0 & 0 \\ -1 & n-1 & & & \\ \vdots & & & & \\ & & & & n-1 \end{vmatrix} \underset{\substack{R_3 - R_4 \\ \vdots}}{\overset{\substack{\text{repeat} \\ R_2 - R_3}}{=}} \begin{vmatrix} n & -n & 0 & \cdots & & 0 \\ 0 & n & -n & 0 & \cdots & \\ & & & & & \\ 0 & \cdots & & & n & -n \\ -1 & -1 & \cdots & & -1 & n-1 \end{vmatrix}$$

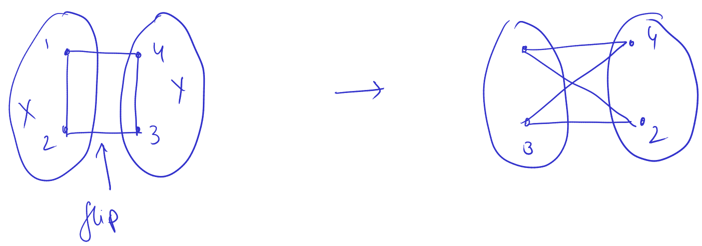
$$C_2 = C_2 + C_1 = \begin{vmatrix} n & 0 & 0 & \\ 0 & n & -n & \\ & & & \\ -1 & -2 & \cdots & \end{vmatrix} \underset{C_{n-2}\,C_{n-1}}{\overset{C_2 + C_3}{\vdots}} = \begin{vmatrix} n & 0 & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & & & \\ & & & & & \cdot \\ & & & & n & -n \\ -1 & -2 & -3 & \cdots & -(n-2) & n-1 \end{vmatrix} \overset{C_{n-1} + C_n = C_n}{=} \begin{vmatrix} n & & & & \\ & n & & & 0 \\ & & \ddots & & \\ & & & n & 0 \\ -1 & -2 & \cdots & & 1 \end{vmatrix}_{n-1\times n-1}$$

$= n^{n-2}$ (lower triangular matrix).

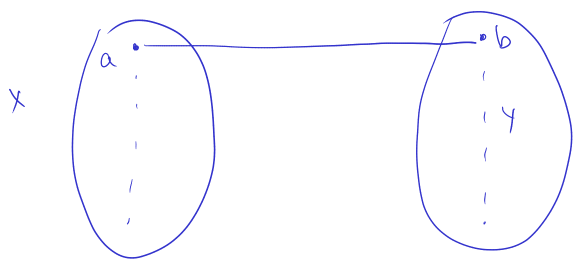
6. (15 points)

Suppose G is a simple graph with no isolated vertices, and let X and Y be a partition of $V(G)$. Define J be the bipartite subgraph of G with partite sets X and Y, where J contains all edges of G that have one endpoint in each partite set.

(a) Suppose $d_J(v) = \frac{1}{2}d_G(v)$ for all $v \in V(J)$, where $d_J(v)$ and $d_G(v)$ denote degrees of the vertex v in the graphs J and G respectively. Show that there is a bipartite subgraph of G with more that $e(J)$ edges. *Hint: Do this first for the graph* C_4*. Let the vertices appear in order as* $1, 2, 3, 4$*. Let the partition be* $X = \{1, 2\}$ *and* $Y = \{3, 4\}$*. Flip one of the edges to achieve a bipartite subgraph with more than* $e(J) = 2$ *edges.*



$e(J) = 4.$



Pick $a \in X$, $b \in Y$ st
ab is an edge
Move a to Y and b to X. Call this graph J'
J loses $|N(a) \cap Y \setminus b|$ edges and gains $|N(a) \cap X|$ edges.

$$\Rightarrow e(J') = e(J) - \left(\frac{d_J(a)}{2} - 1\right) + \frac{d_J(a)}{2}$$

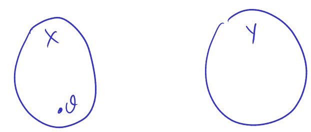
$$= e(J) + 1$$

All the other edges not involving a and b are unaffected.

(b) Show that for every simple graph with no isolated vertices, there is a bipartite subgraph H with $e(H) > e(G)/2$. Recall that in class, we used a swapping method to show there is a bipartite graph with $e(H) \geq e(G)/2$. This problem seeks to improve that method using the observation in part A.

Apply the procedure in Prob 1.3.19 until we arrive at a bipartite graph

st $\quad d_H(v) \geq \dfrac{d_G(v)}{2} \quad \forall\, v \in V(G)$.



(Find v st $\quad d_H(v) < \dfrac{d_G(v)}{2}$

$\iff d_H(v) < d_G(v) - d_H(v)$.

if $v \in X$, move it to Y. We lose $d_H(v)$ edges and gain $d_G(v) - d_H(v)$ which is at least one. Since the total # of edges is bounded, this procedure must stop.)

$$\Rightarrow \sum_v d_H(v) \geq \sum \frac{d_G(v)}{2}$$

$\Rightarrow \quad e(H) \geq \dfrac{e(G)}{2} \qquad$ if $\exists \quad d_H(v) > \dfrac{d_G(v)}{2}$ then we're done since $e(H) > \dfrac{e(G)}{2}$

If not $\quad d_H(v) = \dfrac{d_G(v)}{2} \quad \forall v$. Applying part a finishes the proof.