Math 248: Graph Theory

Midterm

Thursday, March 2nd, 2023

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 11 pages.
- No calculators, phones, electronic devices, or books are allowed during the exam.
- You are allowed the use of a double-sided 3"x5" card for notes.
- Show all work and justify all answers unless otherwise instructed.
- Read the instructions for each problem carefully.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

1. (25 points) For each of the following statements, determine if it is true or false. You do not need to justify your answer.

(a) A vertex in a simple connected graph is a cut-vertex if it is not included in any cycles.

(b) Let G be a digraph such that for every $v \in V(G)$, $d^+(v) + d^-(v) \geq 2$. Then G contains a cycle.

(c) There exists a tree G with at least 2 vertices for which the only automorphism on G is the identity map $\iota : V(G) \to V(G)$.

(d) Given a graph G and a strict subset $W \subset V(G)$, the induced subgraph on W has strictly smaller size than G.

(e) For any positive integer n, there exists a simple graph with radius n and diameter $n + 1$.

Out of questions 2,3,4,5, you only need to solve three questions. Clearly mark which three problems you've chosen to solve, the fourth will not be graded. For each problem you choose to solve, you must solve all parts of that problem.

2. (25 points)

(a) Give a definition of a regular graph.

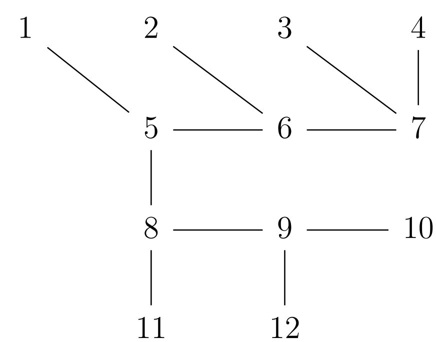
(b) Is $(6, 6, 3, 3, 3, 3, 3, 1, 1, 1)$ a graphic sequence? If so, draw a simple graph which realizes it. If not, explain why not.

(c) **Prove** that if $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is a sequence of *distinct* integers, then $\mathbf{d}$ is not graphic. (Hint: What are the possible vertex degrees for a simple graph on n vertices?)

3. (25 points)

(a) Give a definition of a spanning subgraph of a graph G.

(b) What is the Prüfer code for the following tree?

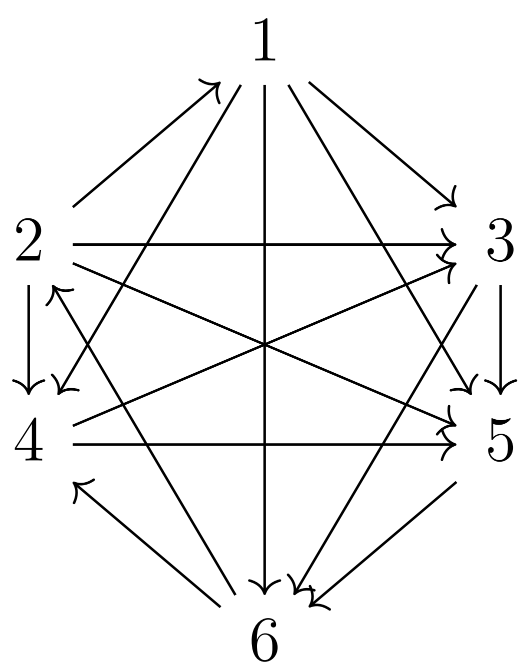


(c) Let G be a simple connected graph. **Prove** that $v \in V(G)$ is a cut-vertex if and only if no spanning tree of G has v as a leaf.

4. (25 points)

(a) Define what it means for a digraph G to be strongly connected.

(b) Find all kings of the following digraph.



(c) **Prove** that there exists an Eulerian tournament on n vertices if and only if n is odd. (Hint: How can you characterize when a digraph will be Eulerian?)

5. (25 points)

(a) Define the graph Laplacian L_G of a loopless graph G.

(b) Consider $G = K_2 + K_3$, a graph on 5 vertices. Find all eigenvectors of L_G corresponding to the eigenvalue 0.

(c) Let G be a loopless graph. **Prove** that 0 is an eigenvalue of L_G with multiplicity at least the number of components of G. (Hint: Can you construct a distinct eigenvector for each component?)

(Note: The multiplicity is actually equal to the number of components, but you don't need to prove that here.)