MATH 240H: Homework 9: Quotient Constructions.
Due on gradescope, 11:59PM on Saturday, April 6

1. Let $X = [0, 2] \times [0, 1]$ be given the subspace topology from the standard topology of $\mathbb{R}^2$. Intuitively this is a strip of paper. In class we considered the equivalence relation of gluing the left and right edges with a twist given by $(0, t) \sim (2, 1 - t)$ for all $0 \leq t \leq 1$. (In general we will not bother listing equivalences of the form $x \sim x$ when describing an equivalence relation and just list the "nontrivial" equivalences in the relation.)The resulting quotient space $X/_\sim$ is the Möbius Band.
(a) Let $L = \{(x, \frac{1}{2})|0 \leq x \leq 2\}$ which is a line segment in X. What is the homeomorphism type of the quotient $L/_\sim$ as a subspace of $X/_\sim$. (No proof needed in this problem, just answer the questions "intuitively" and list a standard space that $L/_\sim$ is homeomorphic to).
(b) Make a model of the Mobius band $X/_\sim$ from paper and cut it along the curve $L/_\sim$. In class we showed abstractly using identification diagrams that the result is a single cylinder. Describe briefly what happens when you actually do the experiment - the results are of course correct but the resulting cylinder will be embedded in $\mathbb{R}^3$ in a nontrivial way.
(c) Further cut the cylinder obtained in the experiment in (b) along its middle "circle". We have seen abstractly that when you cut a cylinder this way you will get two pieces each homeomorphic to a cylinder. Describe briefly what actually happens when you conduct the experiment.

2. Let $RP^n = (\mathbb{R}^{n+1} - \{\hat{0}\})/_\sim$ be the real projective space of $\mathbb{R}^{n+1}$ i.e., the space of lines through the origin in $\mathbb{R}^{n+1}$. Here the equivalence relation $\sim$ is given by declaring $\hat{x}$ to be equivalent to $\hat{y}$ if they lie on the same line through the origin, i.e., if there exists $\lambda \in \mathbb{R} - \{0\}$ such that $\lambda\hat{x} = \hat{y}$. As usual this quotient set is given the quotient topology it inherits from being a quotient of $\mathbb{R}^{n+1} - \{\hat{0}\}$.
(a) Let $S^n \subseteq \mathbb{R}^{n+1} - \hat{0}$ be the sphere of radius 1 with its subspace topology. Show that the equivalence relation $\sim$ restricts to an equivalence relation on S^n where two unit vectors $\hat{u}$ and $\hat{v}$ are equivalent if and only if $\hat{u} = \pm\hat{v}$. (The points $-\hat{u}$ and $\hat{u}$ are said to be antipodal, thus the equivalence relation inherited on the sphere is the one where antipodal points are equivalent.)
(b) Let $i : S^n \to \mathbb{R}^{n+1} - \{0\}$ be the inclusion map $i(x) = x$ for $x \in S^n$. Let $p : \mathbb{R}^{n+1} - \{\hat{0}\} \to RP^n$ be the quotient map in the original description of $\mathbb{R}P^n$. Let $q : S^n \to S^n/_\sim$ be the quotient map for the equivalence relation in

(a). Explain why $f = p \circ i : S^n \to RP^n$ is continuous and why it induces a continuous function $\bar{f} : S^n/_\sim \to RP^n$ using theorems about inducing functions on quotient spaces.
(c) Show that the function $\bar{f} : S^n/_\sim \to RP^n$ is a bijection. (Hint: For an equivalence class $[\hat{x}] \in RP^n$ show that $[\hat{x}] = [\hat{u}]$ where $\hat{u}$ is a vector of length 1.)
(The continuous bijection $\bar{f} : S^n/_\sim \to RP^n$ can be shown to be a homeomorphism using techniques we'll learn in a couple of weeks. Thus RP^n, the space of lines thru the origin of $\mathbb{R}^{n+1}$ can also be viewed as the space created by identifying antipodal points on the n-sphere S^n. Thus you have now verified that we have two equivalent quotient space descriptions of the space $\mathbb{R}P^n$.)

3. (a) Let H be the upper hemisphere of $S^n \subseteq \mathbb{R}^{n+1}$, i.e., $H = \{(x_1, \ldots, x_{n+1}) \in S^n | x_{n+1} \geq 0\}$. Show that H is a closed subspace of S^n.
(b) Let E be the equator of $S^n \subseteq \mathbb{R}^{n+1}$, i.e., $E = \{(x_1, \ldots, x_n, 0) \in S^n\}$ is the subset of vectors of the sphere whose last coordinate is zero. Explain why E is a closed subspace of H and why E is homeomorphic to S^{n-1}.
(c) Let $\sim$ be the equivalence relation on S^n where $\hat{u} \sim -\hat{u}$ as in question 2. Show that this equivalence relation restricts to an equivalence relation on the upper hemisphere H were the only nontrivial equivalences are on the equatorial sphere where antipodal points are identified.
(d) Let $j : H \to S^n$ be the inclusion map. Let $q : S^n \to S^n/_\sim$ and $\pi : H \to H/_\sim$ be quotient maps. Explain why $g = q \circ j$ is continuous and why it induces a continuous map $\bar{g} : H/_\sim \to S^n/_\sim$ given by $\bar{g}([x]) = [j(x)]$ where $[y]$ denotes the equivalence class of y.
(e) Show that the map $\bar{g}$ in (d) is a continuous bijection.
(It can be shown that $\bar{g}$ is actually a homeomorphism. Thus RP^n can be made by starting with H and identifying antipodal points on the equatorial sphere. Thus now we have a third quotient space description of RP^n.)

4. Using the results from problems 2 and 3, it can be shown that a way to construct the projective space RP^2 is to take the upper hemisphere H of the 2-sphere S^2 and identify antipodal points on the equatorial circle. H in turn can be shown to be homeomorphic to a unit closed disk in the plane. Thus a model for RP^2 can be obtained by taking the unit closed disk in the plane and identifying antipodal points on its boundary circle. In this problem you will analyze RP^2 "intuitively" using a model very close to this one.
(a) Let $X = [0, 1] \times [0, 1]$ be given the subspace topology from the stan-

dard topology on the plane. Let $\sim$ be an equivalence relation on X whose nontrivial equivalences are given by $(0,t) \sim (1, 1-t)$ for $0 \leq t \leq 1$ and $(x,0) \sim (1-x, 1)$ for $0 \leq x \leq 1$. In other words the left edge is glued to the right edge with a twist and the top edge is glued to the bottom edge with a twist. Draw a careful picture of the identifications described on X.
(The resulting quotient space can be shown to be homeomorphic to RP^2. In fact the unit closed disk is homeomorphic to the unit square X in such a way that the antipodal identifications on the boundary circle become the identifications described here.)
(b) In your identification model of RP^2 drawn in (a), make a dotted straight line joining $(0,1)$ and $(1,\frac{1}{2})$ and another dotted straight line joining $(0,\frac{1}{2})$ and $(1,0)$. The union L of these two straight lines is a subspace of X. What is the homeomorphism type of $L/_\sim$ as a subspace of $X/_\sim = RP^2$? (No proof needed - just an intuitive answer.)
(c) Cut RP^2 up along the "dotted curve" in (b). You should get three "pieces" though two of them glue together so you only will have two real pieces. Explain informally (using pictures) why one of these pieces is homeomorphic to a Möbius band and the other is homeomorphic to a closed disk in the plane.
(Note: You have now shown that RP^2, the space of lines through the origin in $\mathbb{R}^3$ can be obtained by gluing a disk to a Möbius band along their (common) circle edge.)

5. Let $D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ be the unit closed disk of the plane with its standard subspace topology. Consider the equivalence relation on D where all points on the boundary circle are equivalent, but each point in the interior is only equivalent to itself. What standard simple space is the quotient space $D/_\sim$ homeomorphic to? (No proof needed just do it intuitively as if you were gluing points physically.)

6. (No proofs needed in this question.)
Starting with the space $X = [0,2] \times [0,1]$ draw identifications for an equivalence relation on it so that the resulting quotient space is homeomorphic to:
(a) a cylinder.
(b) a Möbius band.
(c) a torus.
(d) a Klein bottle.
(e) the real projective plane RP^2.

(f) the 2-sphere S^2.

7. [Informal "Cut and Paste" question - answer accordingly]
Let X be a 12-gon in the plane (the boundary as well as the "inside") with the usual Euclidean subspace topology. Going around the boundary counter-clockwise, label the 12 line segments $a, b, a^{-1}, b^{-1}, c, d, c^{-1}, d^{-1}, e, f, e^{-1}, f^{-1}$. We will orient the line segments a, b, c, d, e, f with an arrow pointing along the segment, **counterclockwise** along the boundary while we orient the line segments $a^{-1}, b^{-1}, c^{-1}, d^{-1}, e^{-1}, f^{-1}$ with an arrow pointing along the segment, **clockwise** along the boundary.

(a) Draw a picture of X as described.

(b) Now let Y be the quotient space obtained by identifying the boundary line segments labelled x with their inverse x^{-1} labelled segment according to the assigned arrows for all $x \in \{a, b, c, d, e, f\}$. On the "inside" of the 12-gon no identifications are made. What this means is if you paremetrize the line segments as functions of time $0 \leq t \leq$ then you sweep them out according to the arrows you made in (a) moving with the arrow as time increases. You then identify similar labeled lines by identifying points corresponding to the same time t. Show that all 12 corners of the n-gon become the same point from in the final qoutient space.

(c) Draw two dotted "lines" across the interior of the 12-gon. One joining the end-corners located at the two ends of the $e - f - e^{-1} - f^{-1}$ sequence of boundary edges, and another joining the end-corners located at the two ends of the $a - b - a^{-1} - b^{-1}$ sequence of corners. Explain why these two "lines" are homeomorphic to two circles in the quotient space that meet at a point.

(d) Cut the space along the two dotted circles in (c) to obtain 3 pieces. Explain informally why two of the pieces are homeomorphic to a torus with a hole drilled out (this means you take a "small" chart on the torus and remove an open set corresponding to an open Euclidean disk in that chart - recall the torus is a 2-manifold). Explain informally why the third piece is homeomorphic to a torus with two holes drilled out such where the boundary circles of the holes touch at one point.

(e) Explain informally why the quotient space is homemorphic to the surface of genus 3. (for a picture see Wikipedia entry for "Genus (mathematics)").

8. Using ideas from problem 7, find an equivalence relation on the edges of an octagon (8-gon) such that the resulting quotient space is homeomorphic to the surface of a pretzel (Surface of genus 2). No proofs necessary just carefully draw an octagon and indicate how you are gluing its boundary edges.

9. Consider the quotient map $p : \mathbb{R}^{n+1} - \{0\} \to RP^n$ described earlier in this homework set. Let us denote the equivalence class of the nonzero vector $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} - \{0\}$ by $[x_1, \ldots, x_{n+1}] \in RP^n$.
(a) Let $U_i = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_i \neq 0\}$ for $i = 1, \ldots, n+1$. Show that these U_i are p-saturated open sets that cover $\mathbb{R}^{n+1} - \{0\}$, i.e. whose union is all of $\mathbb{R}^{n+1} - \{0\}$. Then explain why $p(U_i), i = 1, \ldots, n+1$ are open sets in RP^n that cover RP^n.
(b) Show that the function $\lambda_i : U_i \to \mathbb{R}^n$ given by $\lambda_i(x_1, x_2, \ldots, x_{n+1}) = (\frac{x_1}{x_i}, \frac{x_2}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i} \ldots, \frac{x_{n+1}}{x_i})$ is a well-defined, continuous map that is constant on the level sets of p. This implies that λ_i induce well-defined continuous maps $\bar{\lambda}_i : p(U_i) \to \mathbb{R}^n$ where $\bar{\lambda}_i([x_1, \ldots, x_{n+1}]) = \lambda_i(x_1, \ldots, x_{n+1})$.
(c) Show that the map $\psi_i : \mathbb{R}^n \to p(U_i)$ given by

$$\psi_i(y_1, \ldots, y_n) = p(y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n) = [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n]$$

is well defined and continuous. Show that ψ_i is the inverse of λ_i.
(d) Explain why RP^n is a locally n-Euclidean space.