MATH 240H: Homework 8: Manifolds and Topological Groups.
Due at 11:59PM on Saturday, March 30 on Gradescope.com

In this homework set, subsets of $\mathbb{R}^n$ are always equipped with the subspace topology coming from the standard topology of $\mathbb{R}^n$. All products will be given the product topology.

1. (a) Show that if X and Y are 2nd countable topological spaces then $X \times Y$ is also 2nd countable.
(b) Show that if X is a n-dimensional topological manifold and Y is a m-dimensional topological manifold then $X \times Y$ is a $(m+n)$-dimensional topological manifold.

2. Let $F : \mathbb{R}^4 \to \mathbb{R}$ be given by $F(x, y, z, w) = x^3 + y^3 + z^3$.
(a) Find the set of critical points and regular points of F (Note these are both subsets of $\mathbb{R}^4$.)
(b) Find the set of critical values and regular values of F (Note these are both subsets of $\mathbb{R}$.)
(c) Explain why $M = \{(x, y, z, w) \in \mathbb{R}^4 | x^3 + y^3 + z^3 = 10\}$ is a smooth manifold. What is its dimension, i.e., for which k does M "look locally like" $\mathbb{R}^k$?

3. Let $H : \mathbb{R}^4 \to \mathbb{R}^2$ be given by

$$H(x, y, z, w) = (x^3 + y^3 + z^3, x^4 + y^4 + z^4 + w^4).$$

(a) Find the Jacobian matrix $DH_{\hat{x}}$. What are the dimensions of this matrix?
(b) Note that $\hat{x}$ is a regular point as long as $DH_{\hat{x}}$ gives a surjective linear transformation from $\mathbb{R}^4 \to \mathbb{R}^2$, i.e., as long as $DH_{\hat{x}}$ has rank two. Explain why this happens in this case if and only if its two rows are independent vectors in $\mathbb{R}^4$.
(c) Find the critical points and regular points of H (Note these are both subsets of $\mathbb{R}^4$.)
(d) Find the critical values and regular values of H (Note these are both subsets of $\mathbb{R}^2$.)
(e) Explain why $N = \{(x, y, z, w) \in \mathbb{R}^4 | x^3 + y^3 + z^3 = 10, x^4 + y^4 + z^4 + w^4 = 20\}$ is a smooth manifold. What is its dimension?

4. Recall that a topological group $(G, \star)$ is a group where G is a topological space and the group structure maps $\star : G \times G \to G$ and $I : G \to G, I(x) = x^{-1}$ are continuous.
(a) For fixed $g_0 \in G$ we define the left translation by g_0 map, $L_{g_0} : G \to G$, via $L_{g_0}(x) = g_0 \star x$. Show that L_{g_0} is a continuous map. (Hint: Let $c : G \to G$ be the constant map with value g_0 and consider $F : G \to G \times G$ given by $F(x) = (c(x), x)$ and its composition with $\star : G \times G \to G$.)
(b) Let R_{g_0} be the right translation by g_0 map, $R_{g_0} : G \to G$, defined via $R_{g_0}(x) = x \star g_0$. Show that any right or left translation maps L_{g_0}, R_{g_0} are homeomorphisms $G \to G$.
(c) Show that the inversion map $I : G \to G$, $I(x) = x^{-1}$ is a homeomorphism.
(d) Show that if W is open in G then $W^{-1} = \{w^{-1} | w \in W\}$ is also open in G. Show that C is closed in G if and only if C^{-1} is closed in G.
(e) For given fixed $x, y \in G$ show that there is a unique left translation map $L_z : G \to G$ such that $L_z(x) = y$. What is z? (Note this problem shows that topological groups are "homogeneous spaces", i.e., given two points in the space, there is a self-homeomorphism of the space taking one point to the other.)

5. (a) Let $(G_1, \star_1)$ and $(G_2, \star_2)$ be groups. Show that the product set $(G_1 \times G_2)$ is a group under the binary operation $(g_1, g_2) \star (g_1', g_2') = (g_1 \star_1 g_1', g_2 \star_2 g_2')$. $G_1 \times G_2$ is called the product group of G_1 and G_2.
(b) If G_1 and G_2 are topological groups show that the product group $G_1 \times G_2$ is a topological group under the product topology.
(c) If G_1 and G_2 are Lie groups show that the product group $G_1 \times G_2$ is a Lie group.

6. Let $(G, \star)$ be a topological group and let e be its identity element.
(a) Show that if $\{e\}$ is a closed set in G then G is a T_1-space, i.e., a space where every singleton set $\{x\}$ is closed.
(b) Let A, B be subsets of G we define $A \star B = \{a \star b | a \in A, b \in B\}$. Show that $A \star B$ is open in G if either A or B are. (Hint: First show $A \star B = \cup_{a \in A} L_a(B)$ where L_a are left translation maps.)
(c) Let U, V be open subsets of $\mathbb{R}^n$, explain briefly why

$$U + V = \{u + v | u \in U, v \in V\}$$

is open in $\mathbb{R}^n$.
(d) A symmetric neighborhood of the identity is an open set V such that

$e \in V$ and $V^{-1} = V$. Show that if U is any open neighborhood of e in G, then there exists a symmetric neighborhood V such that $V \star V \subseteq U$. (Hint: The map $\star : G \times G \to G$ is continuous and takes (e, e) to e.)
(e) Let $(G, \star) = (\mathbb{R}, +)$ and $A = [1, 2), B = (-2, 3)$. Find the set $A \star B = A + B$ explicitly and also find a symmetric neighborhood V of $e = 0$ such that $V + V \subseteq B$.

7. Let $(G, \star)$ be a topological group.
(a) Let $f : G \times G \to G$ be given by $f(x, y) = xy^{-1}$. Show that f is continuous.
(b) Show that G is a T_1-space if and only if it is a Hausdorff space. (Hint: If $\{e\}$ is closed in G consider $f^{-1}(\{e\})$ in $G \times G$.)

8. It was mentioned in class that the special linear group

$$SL_n(\mathbb{R}) = \{\mathbb{A} | det(\mathbb{A}) = 1\}$$

is a Lie group. In this exercise you will help verify this in the case $n = 2$.
(a) Identify $Mat_2(\mathbb{R})$ as $\mathbb{R}^4$, the determinant map $det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$ is then identified with the map $f : \mathbb{R}^4 \to \mathbb{R}$ by $f(a, b, c, d) = ad - bc$. Find the critical points, regular points, critical values and regular values of this map.
(b) $SL_2(\mathbb{R}) = det^{-1}(\{1\})$. Show that 1 is a regular value of the determinant map in (a) and hence conclude that $SL_2(\mathbb{R})$ is a smooth manifold and hence is a Lie group. What is its dimension?

9. Let $O(n) = \{\mathbb{A} \in Mat_n(\mathbb{R}) | \mathbb{A}\mathbb{A}^T = \mathbb{I}\}$ be the orthogonal group.
Defining $f : Mat_n(\mathbb{R}) \to Sym_n(\mathbb{R})$ given by $f(\mathbb{A}) = \mathbb{A}\mathbb{A}^T$ one can see that $O(n) = f^{-1}(\{\mathbb{I}\})$. It can be shown that $\mathbb{I}$ is a regular value of f and so $O(n)$ is a smooth manifold and hence a Lie group - you may use this freely in this exercise.
(a) The vector space $Sym_n(\mathbb{R}) = \{B \in Mat_n(\mathbb{R}) | B^T = B\}$ is the vector space of symmetric $n \times n$ matrices. (This has to be used as the codomain of f instead of the larger codomain $Mat_n(\mathbb{R})$ as $\mathbb{I}$ can be shown to not be a regular value if the larger codomain is used.) Find a basis of $Sym_2(\mathbb{R})$ as a real vector space and hence find its dimension.
(Hint: $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b? + d?$)
(b) Explain why the dimension of $Sym_n(\mathbb{R})$ as a real vector space for general

$n \geq 1$ is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
(c) Find the dimension of $O(n)$ as a smooth manifold for general $n \geq 1$ using the regular value theorem and the fact that $f(A) = AA^T$ has $\mathbb{I}$ as a regular value when viewed as a map from $Mat_n(R) = \mathbb{R}^{n^2} \to Sym_n(R) = \mathbb{R}^{\frac{n(n+1)}{2}}$.