MATH 240H: Homework 7: Metric Space Analysis.
Due Saturday, March 23, 11:59PM on Gradescope.com

1. Let (X, d) be a metric space. Check that an infinite sequence $\{x_n | n \in \mathbb{Z}_+\}$ converges to α in X if and only if the sequence of distances $d(x_n, \alpha)$ converges to 0 in $\mathbb{R}$ with the standard topology. More succinctly

$$(x_n \to \alpha \in X) \iff (d(x_n, \alpha) \to 0 \in \mathbb{R})$$

2. A normed space is a real vector space V equipped with a "length/norm" function $||\cdot|| : V \to \mathbb{R}$ where $||\hat{v}||$ is interpreted as "the length of $\hat{v}$". The norm is required to satisfy the following properties:
(1) $||\hat{v}|| \geq 0$ and $||\hat{v}|| = 0 \iff \hat{v} = \hat{0}$.
(2) $||\alpha\hat{v}|| = |\alpha|||\hat{v}||$ for all $\alpha \in \mathbb{R}, \hat{v} \in V$.
(3) $||\hat{v} + \hat{w}|| \leq ||\hat{v}|| + ||\hat{w}||$ for all $\hat{v}, \hat{w} \in V$.
(a) Show that in any normed vector space $(V, ||\cdot||)$, the formula $d(\hat{x}, \hat{y}) = ||\hat{x} - \hat{y}||$ defines a metric on V.
(b) In $\mathbb{R}^n$, the Euclidean norm is defined via $||\hat{v}|| = \sqrt{\hat{v}\cdot\hat{v}}$ where $\cdot$ is the standard dot product. Verify properties (1) and (2) for this norm.
(c) To verify property (3), it is easier to first prove the Cauchy-Schwartz inequality:

$$|\hat{v} \cdot \hat{w}| \leq ||\hat{v}||||\hat{w}||$$

Please provide a direct proof of this. (Hint: Reduce to the case $\hat{v}, \hat{w} \neq 0$, let $a = \frac{1}{||\hat{v}||}, b = \frac{1}{||\hat{w}||}$ and use the fact that $||a\hat{v} \pm b\hat{w}|| \geq 0$ which follows as you have verified properties (1) and (2) already.)
(d) Finish the proof that the Euclidean norm is indeed a norm i.e., verify property (3) for this norm. (Hint: Compute $(\hat{u} + \hat{v}) \cdot (\hat{u} + \hat{v})$ and apply (c).)

3. A function $f : (X, d) \to (Y, D)$ between two metric spaces is called α-Holder continuous if there are constants $C, \alpha > 0$ such that

$$D(f(x_1), f(x_2)) \leq Cd(x_1, x_2)^\alpha$$

for all $x_1, x_2 \in X$. The function is called Lipshitz if it is Holder continuous with $\alpha = 1$ i.e.,

$$D(f(x_1), f(x_2)) \leq Cd(x_1, x_2)$$

for all $x_1, x_2 \in X$.
(a) Prove that (thankfully) every Holder continuous function (for any $\alpha > 0$)

is continuous. (Hint: It might be easiest to use that a function is continuous if and only if it is "good with sequences" as we are in a metric setting. Problem 1 might then be helpful.)
(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is Holder continuous with $\alpha > 1$ (where the domain and codomain are given the usual metric $d(x, y) = |x-y|$) then f must be a constant function. (Hint: Use the definition of a derivative in calculus as $f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x}$ to show that a Holder continuous function with $\alpha > 1$ has a very special derivative everywhere. Then use a basic fact from calculus.)
(c) Show that $f : (0, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{x}$ is **not** Lipschitz with respect to the standard metric of the real line, even though it is continuous. (Hint: Compute an explicit formula for $\frac{|f(x)-f(y)|}{|x-y|}$.)

4. Let (X, d) be a metric space. Equip $\mathbb{R}$ with its standard metric thru out this problem.
(a) Show that $d : X \times X \to \mathbb{R}$ is continuous where $X \times X$ is given the product topology.
(b) Show that for fixed $a \in X$, the function $f(x) = d(x, a) : X \to \mathbb{R}$ is Lipschitz and hence continuous. (Hint: Show $|d(x, a) - d(y, a)| \leq d(x, y)$.)
(c) Let $A \subseteq X$, we define $D_A(x) = \inf\{d(x, a) | a \in A\}$. This is called the distance between x and the set A. Note if the infimum is achieved, $D_A(x)$ represents the distance between x and the "closest" point in A. Show that $D_A(x) = 0$ if and only if $x \in \bar{A}$, i.e., x is in the closure of A.
(d) Show that the function $D_A : X \to \mathbb{R}$ satisfies $|D_A(x) - D_A(y)| \leq d(x, y)$ and is hence Lipschitz (and so continuous).
(e) For $\epsilon > 0$, A^ϵ, the ϵ-neighborhood of A, is defined as

$$A^\epsilon = \{x \in X | D_A(x) < \epsilon\}.$$

Explain why A^ϵ is an open set of X containing A.
(f) A G_δ (read "G-Delta") set A in a topological space X is a set which is the countable intersection of open sets, i.e., $A = \cap_{n=1}^\infty U_n$ where U_n is open in X. Prove in a metric space (X, d), that every closed set is a G_δ set. (Hint: First prove that if A is closed then $A = \cap_{n=1}^\infty U_n$ where U_n is the $\epsilon = \frac{1}{n}$ neighborhood of A.)
(g) A F_σ (read "F-Sigma") set B in a topological space X is a set which is the countable union of closed sets, i.e., $B = \cup_{n=1}^\infty C_n$ where C_n is closed in X. Prove in a metric space (X, d), that every open set is a F_σ set.

5. In this question we will consider $\mathbb{R}^\omega$ in the product topology, box topology and uniform topology. Recall the uniform topology is given by the metric $\rho(\hat{x}, \hat{y}) = sup_{n\in\mathbb{Z}_+}\bar{d}(x_n, y_n)$ where $\bar{d}$ is the cutoff metric corresponding to the standard metric $d(x,y) = |x-y|$ on $\mathbb{R}$.
(a) For each of the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$, for which of these three topologies on $\mathbb{R}^\omega$ is the function continuous?
(i) $f(t) = (t, 2t, 3t, 4t, \dots)$
(ii) $g(t) = (t, t, t, t, \dots)$
(iii) $h(t) = (t, \frac{t}{2}, \frac{t}{3}, \frac{t}{4}, \dots)$
(b) In which of the three topologies do the following sequences in $\mathbb{R}^\omega$ converge? What do they converge to if they converge?
(i)
$\hat{w}_1 = (1, 1, 1, 1, \dots)$
$\hat{w}_2 = (0, 2, 2, 2, \dots)$
$\hat{w}_3 = (0, 0, 3, 3, \dots)$
$\dots$
(ii)
$\hat{x}_1 = (1, 1, 1, 1, \dots)$
$\hat{x}_2 = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$
$\hat{x}_3 = (0, 0, \frac{1}{3}, \frac{1}{3}, \dots)$
$\dots$
(iii)
$\hat{y}_1 = (1, 0, 0, 0, \dots)$
$\hat{y}_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots)$
$\hat{y}_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \dots)$
$\dots$
(iv)
$\hat{z}_1 = (1, 1, 0, 0, \dots)$
$\hat{z}_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots)$
$\hat{z}_3 = (\frac{1}{3}, \frac{1}{3}, 0, 0, \dots)$
$\dots$

6. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of the real sequences that are eventually zero, i.e., of the form $(x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$ for some n. Describe the sequences in the closure of $\mathbb{R}^\infty$ in each of the following topologies on $\mathbb{R}^\omega$.
(a) Product Topology.
(b) Box Topology.
(c) Uniform Topology.

7. Let $S_n : \mathbb{R} \to \mathbb{R}$ be given by $S_n(x) = \sum_{k=1}^{n} \frac{1}{2^k} \sin(4^k x)$. We know for any $n \in \mathbb{Z}_+$ this function is a continuous function of x as it is a finite sum of continuous functions.
(a) Show that for any fixed x, the infinite series $\sum_{k=1}^{\infty} \frac{1}{2^k} \sin(4^k x)$ is absolutely convergent and hence convergent. (Recall from calculus, a series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |a_k| < \infty$. Please feel free to use any calculus tests you would like to do this question - just state the test(s) you use.)
(b) From (a), we know $S(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sin(4^k x)$ defines a well defined function $S : \mathbb{R} \to \mathbb{R}$. Show that $|S(x) - S_n(x)| \leq \frac{1}{2^n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}_+$.
(c) Recall on the function space $Func(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ we have the uniform metric given by $\rho(f, g) = sup_{x \in \mathbb{R}} \bar{d}(f(x), g(x))$ where $\bar{d}$ is the cutoff metric corresponding to the standard metric of $\mathbb{R}$. Compute $\rho(S, S_n)$ and use this to explain why S_n converges uniformly to S on the real line.
(d) Explain why $S : \mathbb{R} \to \mathbb{R}$ is continuous.
(e) Compute the derivative $S_n'(0)$. Does $\lim_{n\to\infty} S_n'(0)$ exist in $\mathbb{R}$?

8. A ultrametric d on a set X is a function $d : X \times X \to \mathbb{R}$ such that
(1) $d(x, y) \geq 0$ and $(d(x, y) = 0 \iff x = y)$
(2) $d(x, y) = d(y, x)$
(3) $d(x, y) \leq max(d(x, z), d(z, y))$ for all $x, y, z \in X$.
Notice only the last property is different than the property of a usual metric.
(a) Show that every ultrametric is a metric, i.e. the last property implies the triangle inequality. Give an example to show the converse is not true. (Sometimes an ultrametric is called a non-Archimedean metric).
(b) Show that in an ultrametric space, if $d(x, z) < d(y, z)$ then $d(y, z) = d(x, y)$. (Pictorially this means that the "triangle" made by vertices x, y, z has two sides of the same length and the third side is shorter or the same length also.)
(c) Show that in an ultrametric space, that if two open balls of positive radius have non empty intersection, then they must be nested, i.e. one ball must be contained in the other ball. (Hint: First show that in ultrametric spaces, if $z \in B_d(x, r)$ then $B_d(x, r) = B_d(z, r)$ i.e. that every point inside an open ball can be used as the center of the ball! This is definitely not true in most metrics but only true for ultrametrics!)

9. Fix a prime p. One learns in elementary arithmetic that any positive integer n has a unique base p-expansion $n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k$

where $a_j \in \{0, 1, \ldots, p-1\}$. For example the base 3 expansion of 10 is $10 = 1 + 0 \cdot 3 + 1 \cdot 3^2$.
(a) Find the base $2, 3, 5, 7$ and 11 expansions for $n = 24$.
(b) Let $\mathbb{Z}$ denote the integers. For any integer n define the "p-adic norm" of n, denoted by $||n||_p$, as follows: $||0||_p = 0$, $||n||_p = (\frac{1}{p})^k$ where k is the smallest nonnegative integer index such that $a_k \neq 0$ in the base-p expansion of

$$|n| = a_0 + a_1 p + \cdots + a_m p^m.$$

In others words $||n||_p = (\frac{1}{p})^k$ if p^k divides n but p^{k+1} does not divide n. (Recall for any two nonzero integers m, n we say m divides n if $\frac{n}{m}$ is also an integer.) Show that this "norm" satisfies properties (1) and (3) in exercise 2 and also the property $||mn||_p = ||m||_p ||n||_p$ for any nonzero m, n. In fact show that $||x + y||_p \leq max(||x||_p, ||y||_p)$ in order to check (3).
(c) Find $||24||_p$ for the primes $p = 2, 3, 5, 7, 11$.
(d) Show that $d_p(x, y) = ||x - y||_p$ is an ultrametric on $\mathbb{Z}$. This is called the p-adic metric on $\mathbb{Z}$ (this metric can also be extended to the bigger set of rational numbers after certain modifications).
(e) Show that for any of the p-adic metrics on $\mathbb{Z}$, that the sequence $a_n = n!$ converges to 0.