MATH 240H: Homework 6: Continuous functions and Homeomorphisms.
Due Saturday, March 2 at 11:59PM on gradescope.com

1. In this question X, Y, Z, W are topological spaces.
(a) Check that if Y has the trivial (indiscrete) topology then any function $f : X \to Y$ is continuous.
(b) Check that if W has the discrete topology then any function $g : W \to Z$ is continuous.

2. The sphere S^{n-1} is defined as the set

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 = 1\}$$

endowed with the subspace topology coming from $\mathbb{R}^n$ with its standard topology.
(a) Let $p : \mathbb{R}^n \to \mathbb{R}$ be the function $p(x_1, \dots, x_n) = x_1^2 + \cdots + x_n^2$. Explain how the sphere can be written as a preimage set of p and use this to explain why S^{n-1} is closed in $\mathbb{R}^n$.
(b) Let us now consider the circle $S^1 \subseteq \mathbb{R}^2$. Consider the subset $A = \{(x, y) \in S^1 | y > 0\}$. Explain why A is open in S^1 and give a homeomorphism between A and the interval $(-1, 1)$ topologized as a subspace of $\mathbb{R}$. Briefly justify why the function you gave is a homeomorphism.
(c) Using ideas similar to (b), find a finite number of open sets $A_1, \dots, A_n$ of S^1 such that $S^1 = A_1 \cup \cdots \cup A_n$ and each open set A_i is homeomorphic to $(-1, 1)$ and hence to $\mathbb{R}$ in the standard topology. (Thus you have shown that around every point in S^1, there is an open nhd. that "looks like" i.e., is homeomorphic to, $\mathbb{R}$. This is why we view the circle as "1-dimensional" and why we denote the circle by S^1. One can show that $S^{n-1} \subseteq \mathbb{R}^n$ "looks locally like" $\mathbb{R}^{n-1}$ in a similar way.)

3. In this question all subsets of $\mathbb{R}^n$ will be given the subspace topology coming from the standard topology of $\mathbb{R}^n$. All products will be given product topologies.
(a) Show that $\mathbb{R}^n - \{0\}$ is homeomorphic to $(0, \infty) \times S^{n-1}$ for all $n \geq 1$.
(b) Show that the cylinder $Y = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1, 1 \leq z \leq 2\} \subseteq \mathbb{R}^3$ is homeomorphic to $S^1 \times [1, 2]$. Use this to explain why Y is also homeomorphic with the space $Z = \{(x, y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 \leq 2\}$.

4. (a) Let $D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the "dot-product" function $D(\hat{x}, \hat{y}) = \hat{x} \cdot \hat{y} = \sum_{k=1}^{n} x_k y_k$. Explain briefly why D is continuous.
(b) Consider the n-fold product space $S^{n-1} \times \cdots \times S^{n-1}$. Let X be the subspace given by

$$X = \{(\hat{v}_1, \dots, \hat{v}_n) | \hat{v}_i \cdot \hat{v}_j = \delta_{i,j}\}$$

where $\delta_{i,j}$ is the Kronecker delta function i.e., $\delta_{i,j} = 1$ when $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.
Show that X is a closed subspace of $S^{n-1} \times \cdots \times S^{n-1}$.
(c) Let $O(n)$ be the set of $n \times n$ orthogonal matrices. Recall these are the $n \times n$ real matrices whose rows (or columns) form an orthonormal set. Topologize $O(n)$ as a subspace of $Mat_n(\mathbb{R}) = \mathbb{R}^{n^2}$.
Define $\theta : O(n) \to S^{n-1} \times \cdots \times S^{n-1}$ by $\theta(\mathbb{A}) = (\hat{v}_1, \dots, \hat{v}_n)$ where $\hat{v}_i$ is the ith column of $\mathbb{A}$.
Explain carefully why θ induces a homeomorphism between $O(n)$ and the topological space X discussed in (b).

5. A subset A of a topological space X is called dense if $\bar{A} = X$.
Suppose $f, g : X \to Y$ are two continuous functions and Y is Hausdorff and A is a dense subset of X. Show $f|_A = g|_A \to f = g$, i.e., two continuous functions which agree on a dense subset must agree everywhere. (Hint: Let $T = \{x \in X | f(x) = g(x)\}$ and show T is closed in X by considering the function $F = (f, g) : X \to Y \times Y$.)

6. A function $f : A \to B$ is called a topological embedding if it induces a homeomorphism $A \to f(A)$ where $f(A)$ is given the subspace topology from B. Thus embeddings are maps which induce homeomorphisms from their domain to their image sets but their image sets do not have to be the whole of their codomain.
(a) Let X, Y be topological spaces and $X \times Y$ their product. Show for fixed $x_0 \in X$, the function $f : Y \to X \times Y$ given by $f(y) = (x_0, y)$ is an embedding. (This embedding is called a "vertical slice embedding" which makes sense if you think of $X \times Y$ as a box in the $x - y$ plane.)
(b) A function $X \times Y \to Z$ is called continuous in each variable separately if for each fixed $x_0 \in X$, the function $h : Y \to Z$ given by $h(y) = F(x_0, y)$ is continuous and if for each fixed $y_0 \in Y$, the function $g : X \to Z$ given by $g(x) = F(x, y_0)$ is continuous. Show that if $F : X \times Y \to Z$ is continuous then it is continuous in each variable separately.

7. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x,y) = \begin{cases} \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

(a) Show that F is continuous in each variable separately. (See problem 6 for definition)
(b) Compute an explicit formula for $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x,x)$.
(c) Show that g defined in (b) is not continuous and use it to show F is not continuous.
(Thus you have shown that functions that are continuous in each variable separately need not be continuous in general.)

8.
(a) Is the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \begin{cases} x^2 \text{ if } x \leq 1 \\ 2 - x \text{ if } x \geq 1 \end{cases}$$

continuous? If so give a quick justification, if not explain why not.
(b) Is the function $G : \mathbb{R} \to \mathbb{R}$ given by

$$G(x) = \begin{cases} x \text{ if } x < 1 \\ 1 - x \text{ if } x \geq 1 \end{cases}$$

continuous? If so give a quick justification, if not explain why not.
(c) Is the function G from (b) continuous as a function $G : \mathbb{R}_\ell \to \mathbb{R}$? Here $\mathbb{R}_\ell$ is the real line equipped with the lower limit topology.
(d) Find an example of a function $F : \mathbb{R} \to \mathbb{R}$ and two sets A, B such that $\mathbb{R} = A \cup B$, $F|_A : A \to \mathbb{R}, F|_B : B \to \mathbb{R}$ are continuous but $F : \mathbb{R} \to \mathbb{R}$ is **continuous at no point** on the real line.

9.
Consider $\mathbb{R}^{[0,1]} = \{f : [0,1] \to \mathbb{R}\}$ equipped with the product topology. Recall this is the topology of pointwise convergence i.e. $f_n \to f \in \mathbb{R}^{[0,1]}$ (with product topology) if and only if $f_n(a) \to f(a)$ in $\mathbb{R}$ (with the standard topology) for all $a \in [0,1]$. For each of the following sequences, find the pointwise

limit of the sequence **if it exists**:
(a) $g_n(x) = \frac{1}{n}x$ for all $n \in \mathbb{Z}_+, x \in [0,1]$.
(b) $h_n(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ for all $n \in \mathbb{Z}_+, x \in [0,1]$.

10.
Let $\mathbb{R}^n_Z$ denote $\mathbb{R}^n$ equipped with the Zariski topology where the closed sets are common zero sets of families of polynomials (in multiple variables).
(a) Show that any polynomial $p(x_1, \ldots, x_n)$ induces a continuous map $p : \mathbb{R}^n_Z \to \mathbb{R}^1_Z$. Your proof should hold in cases when $n \geq 2$ also.
(b) Show that $f(x) = \sin(x)$ does **not** give a continuous map $f : \mathbb{R}^1_Z \to \mathbb{R}^1_Z$.