MATH 240H: Homework 5: Hausdorff spaces and Closures.
Due Saturday, Feb 24, 11:59PM on Gradescope.com

1. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$ (with the product topology).

2. Show that a subspace of a Hausdorff space is Hausdorff.

3. Show that if X, Y are Hausdorff spaces then $X \times Y$ is Hausdorff.

4. Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) | x \in X\}$ is closed in $X \times X$.

5. Recall from class that the Zariski Topology on $\mathbb{R}^n$ is the topology where the **closed** sets are of the form $Z(\mathfrak{P}) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | p(x_1, \ldots, x_n) = 0 \text{ for all } p \in \mathfrak{P}\}$ where $\mathfrak{P}$ is a collection of real polynomials. If the collection is just a single polynomial p we will write $Z(p) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | p(x_1, \ldots, x_n) = 0\}$ for the zero set of p. Recall we also saw that the Zariski Topology on $\mathbb{R}^1$ is the same as the cofinite (finite complement) topology. The purpose of this exercise is to study this topology a bit in $\mathbb{R}^2$.
(a) Draw pictures of the Zariski closed sets

$$C_1 = Z(x^2 + y^2 - 1)$$

and

$$C_2 = Z(xy - 1)$$

in $\mathbb{R}^2$. Give a single polynomial $p(x,y)$ such that $C_1 \cup C_2 = Z(p)$.
(b) Now consider the collection $\mathfrak{P} = \{x-2, y-3\}$ of two polynomials. What is $Z(\mathfrak{P}) = Z(x-2, y-3)$ geometrically?
(c) A topological space X is called a T_1-space if every singleton set $\{x\}$ is closed in X. Explain why $\mathbb{R}^n$ with the Zariski Topology is a T_1-space.
(d) Show that $\mathbb{R}^1$ with the Zariski Topology gives an example of a T_1 space which is not Hausdorff.
(Note: The book shows that every Hausdorff space is T_1, this shows that the converse is not true in general.)

6. Given a topological space X and $A \subseteq X$, we say A is **dense** in X if $\bar{A} = X$ where $\bar{A}$ is the closure of A. Show that in the standard topology on the real

line $\mathbb{R}$, both the set of rationals $\mathbb{Q}$ and the set of irrationals $\mathbb{R} - \mathbb{Q}$ are dense.

7. Let A, B and A_α be subsets of a topological space X. Prove the following facts about closures:
(a) If $A \subseteq C$, C closed in X then $\bar{A} \subseteq C$. Thus $\bar{A}$ is the smallest closed set of X containing A.
(b) If $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$.
(c) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
(d) $\overline{\cup_\alpha A_\alpha} \supseteq \cup \bar{A}_\alpha$. Give an example where equality fails.
(e) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. Give an example where equality fails.

8. Let $\bar{S}_\omega = \mathbb{Z}_+ \cup \{\omega\}$ be the set of positive integers together with the first infinite ordinal ω ordered so ω is the largest element and the positive integers has its usual ordering. This is a well-ordered set and we give it the order topology. Show that $\omega \in \bar{\mathbb{Z}}_+$, i.e., ω is in the closure of the set of positive integers in this space.

9. [Kuratowski's Theorem]. Let X be a topological space. Notice the closure operation defines a function $C : P(X) \to P(X)$ on the power set of X, where $C(A) = \bar{A}$. Similarly the complement operation defines a function $M : P(X) \to P(X)$, where $M(A) = X - A$. Kuratowski studied the effects of applying these two operations on a given set A in various orders and in this exercise we will attempt to do the same!
(a) Explain why $M \circ M = Id_{P(X)}$ and $C \circ C = C$ where $\circ$ is composition of operations.
(b) Because of (a), it is clear that if one is going to perform operations M and C in various orders to a set A, one should only consider orders of operations that alternate between applying C and M to get anything new. Let

$$A = (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, 5] \cap \mathbb{Q})$$

be a subset of $\mathbb{R}$ with the standard topology. Show that under alternate use of closure and complement operations you can generate 14 distinct sets from A. (Hint: Compute $C(A), M(C(A)), C(M(C(A))$ etc. until you get a repeat. Then do the same for $M(A), C(M(A))$ etc.)
(c) [BONUS - OPTIONAL - 1 BONUS POINT] Kuratowski proved that in general the maximum number of different sets one can generate from a set A in a topological space X under the closure and complement operations is

14. A key part of the proof is to prove that $C \circ M \circ C \circ M \circ C \circ M \circ C \circ M = C \circ M \circ C \circ M$ in general. Provide a proof. You may look up stuff for help but write it up in your own words.

10. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Show that f is continuous at no point of the real line.
(Hint: If $x \in \mathbb{Q}$, then $f(x) = 1$. Consider $V = (0.5, 1.5)$ as an open nhd. of $f(x)$ and explain why no open nhd. U of x has $f(U) \subseteq V$. Then do a similar thing for irrational points.)
(b) Let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} |x| \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Show that g is continuous at only one point on the real line. Which point is it?