MATH 240H: Homework 4: Product and Subspace Topologies.
Due Saturday Feb 17, 11:59PM on Gradescope.com

1. Recall that the basis $\mathfrak{B}$ of all open disks of positive radius generates the standard topology on the plane $\mathbb{R}^2$. Fix a "cutoff" real number $\epsilon > 0$ and look at the basis $\mathfrak{B}'$ of all open disks of positive radius r such that $0 < r < \epsilon$. Notice that $\mathfrak{B}' \subset \mathfrak{B}$ is a smaller basis. Explain why both $\mathfrak{B}'$ and $\mathfrak{B}$ generate the same topology on the plane $\mathbb{R}^2$. (Note this shows that restricting the basis of open disks to just small open disks below a certain cutoff size ϵ has no effect on the final topology generated.)

2. Let $\mathbb{R}_\ell$ denote the real line with the lower limit topology and $\mathbb{R}$ denote the real line with the standard topology.
(a) Draw a typical basis element of $\mathbb{R}_\ell \times \mathbb{R}$ in the resulting product topology.
(b) Consider $A = \{(x, 0)|x \in \mathbb{R}\}$, i.e., the x-axis in the plane. What topology does A inherit as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$. Draw pictures showing the basis of this subspace topology via intersections of basis elements of $\mathbb{R}_\ell \times \mathbb{R}$ with A.
(c) Identifying A with the real line (via vector space isomorphism say), the subspace topology you found in (b) can be seen to be equivalent to one of the following topologies: standard, trivial, discrete, lower limit. Which one is it? (Note an upper limit topology is basically the same as the lower limit topology if you turn the line around so we won't distinguish those in these questions.)
(d) Consider $B = \{(0, y)|y \in \mathbb{R}\}$, i.e., the y-axis in the plane. What topology does B inherit as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$? Draw pictures to justify your answer and also identify it as in (c).
(e) Now let L_θ be the line thru the origin which makes an angle of θ radians with the x-axis where $0 \leq \theta < \pi$. Use pictures to describe the subspace topology inherited by L_θ and describe it as in part (c).
(Hint: Different placements of basis elements of $\mathbb{R}_\ell \times \mathbb{R}$ intersect the line L_θ differently but in the end, you should look for the topology generated by all possible intersections. The "finest" possible intersections are then the ones that will matter. Also note your answer might depend on θ).

3. Let $\mathbb{R}_\ell$ denote the real line with the lower limit topology.
(a) Draw a typical basis element of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.
(b) Let L_θ be the line thru the origin in the plane which makes an angle of θ radians with the x-axis as in question 2. Determine the subspace topology

induced on L_θ as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.
(Note: Be careful as the answer will depend on θ. You might want to consider lines of positive and negative slope separately. Make sure to consider all possible intersection scenarios of basis elements of $\mathbb{R}_\ell \times \mathbb{R}_\ell$ with L_θ to be able to determine the topology generated correctly.)

4. Let S^1 denote the unit circle in the plane. Topologize the plane as $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Consider the (nonstandard) subspace topology induced on the circle S^1 from $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Using pictures of intersections of basis elements of $\mathbb{R}_\ell \times \mathbb{R}_\ell$ with the circle, decide for what $x \in S^1$ is the singleton set $\{x\}$ open in this nonstandard subspace topology. (Hint: Consider the circle quadrant by quadrant.)

5. Let $A \subseteq B \subseteq X$. Let τ be a topology on X and $\tau|_B$ the induced subspace topology on B. Now note A can be topologized as a subspace of X via $\tau|_A$ or it can be topologized as a subspace of B via $(\tau|_B)|_A$. Show that these topologies are always the same, i.e., A inherits the same topology as a subspace of B as it does as a subspace of X.

6. Show that the collection of rational boxes $\mathfrak{B} = \{(a, b) \times (c, d) | a < b, c < d, a, b, c, d \in \mathbb{Q}\}$ is a countable collection and is a basis for a topology on $\mathbb{R}^2$. Show that it generates the standard topology on the plane $\mathbb{R}^2$, i.e. the same topology that the basis of all open boxes (with or without the rationality conditions) does. (Thus $\mathbb{R}^2$ with the standard topology has a countable basis. Such spaces are called **second countable**.)

7. A map $f : X \to Y$ between topological spaces is called an **open map** if U open in X implies $f(U)$ open in Y. Let A, B be topological spaces, $A \times B$ topologized via the product topology and $\pi : A \times B \to A$ be the canonical projection map to the first coordinate, i.e., $\pi(a, b) = a$ for all $(a, b) \in A \times B$. Show that π is an open map.
(Hint: First consider the case of basis open sets and then argue for general open sets U.)

8. Consider two topologies on the plane. The first τ_{dict}, the dictionary order topology on the plane. The second is the product topology τ' of $\mathbb{R}_d \times \mathbb{R}$ where $\mathbb{R}_d$ denotes the real line with the discrete topology and $\mathbb{R}$ denotes the real line with the standard topology.
(a) Draw pictures of the basis elements in these two topologies on the plane.

(b) Show that these two topologies are in fact the same topology on the plane.

The definitions recalled next will be used in question 9. Recall given a topological space X and a subset A we define:
The interior of A:
$Int(A) = \{x \in X | \exists U, U \text{ open in } X, a \in U \subseteq A\}$.
The exterior of A:
$Ext(A) = \{x \in X | \exists U, U \text{ open in } X, a \in U \subseteq X - A\} = Int(X - A)$.
The boundary of A:
$\partial(A) = \{x \in X | \text{ all open nhds. } U \text{ of } x \text{ have } U \cap A \neq \emptyset, U \cap (X - A) \neq \emptyset\}$.
The closure of A:
$\bar{A} = Int(A) \cup \partial(A) = A \cup \partial(A) = X - Ext(A)$.
$= \{x \in X | \text{ all open nhds. } U \text{ of } x \text{ intersect } A \text{ nontrivially}\}$.
The limit points of A:
$A' = \{x \in X | \text{ all open nhds. } U \text{ of } x \text{ intersect } A \text{ in a point besides } x\}$.

9. Consider $A = (0, 1] \cup \{3\} \cup (4, 7) \subseteq \mathbb{R}$.
Find $Int(A), Ext(A), \partial(A), \bar{A}, A'$ in each of the following topologies of $\mathbb{R}$:
(a) Standard Topology.
(b) Lower Limit Topology.
(c) Discrete Topology.
(d) Finite complement Topology.
(e) Topology with basis $\{(-\infty, a) | a \in \mathbb{R}\}$.

10. Consider the following four real sequences:
$x_n = \frac{1}{n}$, $y_n = 1 - \frac{1}{n}$, $z_n = n$, $h_n = 1$ for all n. For each of these four sequences determine all limits, if any exist, in the following topologies of $\mathbb{R}$:
(a) Standard Topology.
(b) Lower Limit Topology.
(c) Discrete Topology.
(d) Finite complement Topology.
(e) Topology with basis $\{(-\infty, a) | a \in \mathbb{R}\}$.