MATH 240H: Homework 3: Topological Spaces, Basis and Ordered Sets.
Due Saturday, Feb 10, 11:59PM

1. Let X be a nonempty set.
(a) Explain why $\mathfrak{B}_1 = \{\{x\}|x \in X\}$, the collection of singleton sets, is a basis for a topology on X. What topology does it generate?
(b) Explain why $\mathfrak{B}_2 = \{X\}$ is a basis for a topology on X. What topology does it generate?

2. Consider the following topologies on the plane $\mathbb{R}^2$.
τ_1 the topology arising from the basis of open disks of positive radius. This is called the standard topology of the plane.
τ_2 the topology arising from the basis
$\mathfrak{B}_2 = \{[a, b) \times [c, d)|a < b, c < d, a, b, c, d \in \mathbb{R}\}$.
τ_3 the topology arising from the basis
$\mathfrak{B}_3 = \{[a, b] \times [c, d]|a < b, c < d, a, b, c, d \in \mathbb{R}\}$.
τ_4 the topology arising from the basis
$\mathfrak{B}_4 = \{(a, b] \times (c, d]|a < b, c < d, a, b, c, d \in \mathbb{R}\}$.
Let
$A = \{(x, y) \in \mathbb{R}^2|xy > 1, x > 0, y > 0\}$,
$B = \{(x, y) \in \mathbb{R}^2|xy \geq 1, x > 0, y > 0\}$,
$C = \{(x, y) \in \mathbb{R}^2|xy > 1, x < 0, y < 0\}$,
$D = \{(x, y) \in \mathbb{R}^2|xy \geq 1, x < 0, y < 0\}$.
For each of the sets A, B, C, D draw a rough sketch of the region and state which of the four topologies the set is open in and which ones it is not open in.

3. (a) Let r be a rational number. Show that for any $\epsilon > 0$, there exists rational numbers q_1, q_2 such that $r - \epsilon < q_1 < r < q_2 < r + \epsilon$. (Hint: $lim_{n\to\infty}\frac{1}{n} = 0$).
(b) Explain why any real number whose decimal expansion is eventually all zeros or all nines is a rational number.
(c) Let ξ be an irrational real number. For any $\epsilon > 0$, show that there exists rational numbers q_1, q_2 such that $\xi - \epsilon < q_1 < \xi < q_2 < \xi + \epsilon$.
(Notice that from (a) and (c), you have shown any real number has rational numbers arbitrarily close below and above it.)

4. (a) The basis $\mathfrak{B}_1 = \{(a, b)|a < b, a, b \in \mathbb{R}\}$ of all open intervals is a basis for the standard topology on the real line $\mathbb{R}$.

Consider the smaller basis $\mathfrak{B}_2 = \{(a,b)|a < b, a, b \in \mathbb{Q}\}$ of all open intervals with rational endpoints.
(a) Show that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ determine the same topology.
(b) Show that $\mathfrak{B}_1$ is an uncountable set. (Hint: Define a map $\mathfrak{B}_1$ onto $\mathbb{R}$).
(c) Show that $\mathfrak{B}_2$ is a countable set. (Hint: Define a bijection between $\mathfrak{B}_2$ and a subset of $\mathbb{Q} \times \mathbb{Q}$.)
(Notice then that in this example you have seen an example showing that the cardinality of a basis for a given topology is not unique (unlike basis for vector spaces). Also you have shown that the standard topology on $\mathbb{R}$ does have a countable basis. A topological space which has a countable basis for its topology is called **second countable**.)

5. Show that the basis $\{[a,b), a < b, a, b \in \mathbb{Q}\}$ generates a different topology on $\mathbb{R}$ than the basis $\{[a,b), a < b, a, b \in \mathbb{R}\}$. Which one is finer?

6. Show that the dictionary order topology on the plane $\mathbb{R}^2$ is strictly finer than the standard topology coming from the basis of open disks. (Hint: Consider dictionary order open intervals of the form $(x \times y, x \times z)$ where we used $x \times y$ to denote the Cartesian product to not confuse it with the interval notation.)

7. Recall for a nonempty subset S of the real numbers $\mathbb{R}$, we define the supremum, denoted $\sup(S)$, as either ∞ if S is not bounded above or as the least upper bound of S if it is bounded above.
Similarly we define the infimum, denoted $\inf(S)$, as either $-\infty$ if S is not bounded below or as the greatest lower bound of S if it is bounded below.

We say S has a maximum $\sup(S)$ only if $\sup(S) \in S$ and otherwise say S has no maximum. Similarly we say S has a minimum $\inf(S)$ only if $\inf(S) \in S$ and otherwise say S has no minimum.

For each of the following subsets of $\mathbb{R}$ in the standard ordering $<$, determine their minimum, maximum, supremum and infimum if they exist.
(a) $A = [0,1)$.
(b) $B = \{\frac{1}{n}|n \in \mathbb{Z}_+\}$.
(c) $C = \mathbb{Z}_+$.

8. An ordered set $(X, <)$ is said to have the least upper bound property if every nonempty subset S with an upper bound in X has a least upper bound. Similarly it is said to have the greatest lower bound property if every

nonempty subset S with a lower bound in X has a greatest lower bound. Show that if $(X, <)$ has the least upper bound property then it automatically has the greatest lower bound property. (Hint: Consider the set of lower bounds of a given subset S.)

9. Recall a well ordered set is an ordered set $(X, <)$ where every nonempty subset S has a minimum element $s \in S$.
(a) Show that a well ordered set has the least upper bound property.
(b) Show that in a well ordered set, every element (except the largest if one exists) has an immediate successor.

10. Show that if $(X, <_X)$ and $(Y, <_Y)$ are well-ordered sets then $(X \times Y, <_{dict})$ is also well-ordered. Here $<_{dict}$ is the dictionary ordering.

11. Two ordered sets $(X, <_X)$ and $(Y, <_Y)$ have the same order type if there is a bijection $X \to Y$ which preserves order i.e., such that $x_1 < x_2 \to f(x_1) < f(x_2)$. Show the following:
(a) If X and Y have the same order type and X has a smallest element then so does Y.
(b) If $f : X \to Y$ is an order preserving bijection and a is an immediate predecessor of b in X, then $f(a)$ is an immediate predecessor of $f(b)$ in Y.
(c) Show that an order preserving bijection $f : X \to Y$ will induce a bijection between the set of elements in X with an immediate predecessor and the set of elements in Y with an immediate predecessor.

12. Consider the ordered sets $(\mathbb{Z}_+, <)$, $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$, all with the ordering coming from the usual ordering of real numbers. In addition consider the sets $\{0, 1\} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \{0, 1\}$ with dictionary orderings and where $\{0, 1\}$ is ordered by $0 < 1$. Only two of these five sets have the same order type - thus they determine 4 distinct order types. Using ideas similar to those studied in Problem 11, for each pair of these 5 sets, decide if they have the same order type or not. If you determine they do not, just state a short reason why not and if you state that they do, provide an explicit description of the order-preserving bijection. Notice that all 5 sets are countably infinite so these give examples of ordered sets with bijections between them but no order-preserving bijection between them.