MATH 240H: Homework 2: Functions and Cardinality.
Due Saturday, Feb 3, 11:59PM on Gradescope.com

1. Let $f : A \to B$ and $g : B \to C$ be functions and $g \circ f : A \to C$ be their composition.
Show that for any $S \subseteq C$ we have $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$ as an equality of preimage sets.

2. In class, we saw that if $f : A \to B$ and $g : B \to A$ have $g \circ f = 1_A$ then f must be injective and g must be surjective. Show by constructing an explicit example that f need not be surjective and g need not be injective in this situation. (Hint: examples using finite sets or "calculus functions" exist.)

3. Recall given a set X, $X^n = X \times \cdots \times X = \{(x_1, x_2, \ldots, x_n) | x_i \in X \text{ for all } 1 \leq i \leq n\}$ is the n-fold Cartesian product of X with itself and $X^\omega = \{(x_1, x_2, \ldots) | x_i \in X \text{ for all } i \in \mathbb{Z}_+\}$ is the set of infinite sequences in X. Assume $X \neq \emptyset$ for this exercise. $m, n \in \mathbb{Z}_+$. Let $\alpha \in X$ be a specific element that you can use if you need to:
(a) If $m \leq n$, write down an explicit injective map $f : X^m \to X^n$.
(b) Find an explicit bijective map $X^m \times X^n \to X^{n+m}$.
(c) Find an explicit injective map $X^n \to X^\omega$.
(d) Find an explicit bijective map $X^n \times X^\omega \to X^\omega$.
(e) Find an explicit bijective map $X^\omega \times X^\omega \to X^\omega$.
(f) Find an explicit bijective map $(X^\omega)^n \to X^\omega$.
(Hint: You might try to think about these in the case $X = \mathbb{R}$ first before doing it for a general set if you find that helpful.)

4. The following subsets of $\mathbb{R}^\omega$ can be written as Cartesian products $\times_{n=1}^{\infty} A_n$ where $A_n \subseteq \mathbb{R}$ for all $n \in \mathbb{Z}_+$. Find the sets A_n in each of the following examples. We will use the notation $\hat{x}$ as shorthand for $(x_1, x_2, \ldots)$.
(a) $\{\hat{x} | x_i \text{ is an integer for all } i \in \mathbb{Z}_+\}$.
(b) $\{\hat{x} | x_i \geq i \text{ for all } i \in \mathbb{Z}_+\}$.
(c) $\{\hat{x} | x_i \text{ is an integer for all } i \geq 100\}$.

5. Let X be a set and $A \subseteq X$. The characteristic function of A is a function $\chi_A : X \to \{0, 1\}$ such that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Let $P(X)$ be the power set of X, i.e., the set of all subsets of X. Let $Func(X, \{0, 1\}) = \{f : X \to \{0, 1\}\}$ be the set of all functions from X to

$\{0,1\}$. Define $\theta : P(X) \to Func(X, \{0,1\})$ via $\theta(A) = \chi_A$. Show that θ is a bijection.

6. Let X be a nonempty set. Write down an explicit bijection between $Func(\{1,\ldots,n\}, X) = \{f : \{1,\ldots,n\} \to X\}$ and X^n.

7. Let X be a nonempty set. Write down an explicit injection $h : X \to P(X)$. Notice h will induce a bijection between X and its image $h(X)$ which is a subset of $P(X)$.

8. Let X be a finite nonempty set and let $|X|$ denote its cardinality. It is a fact that for finite sets X, Y, $|X \times Y| = |X||Y|$ and you may use that in this problem.
(a) Show that $|P(X)| = 2^{|X|}$. (Hint: Exercise 5 and 6 may be helpful.)
(b) Explain why part (a) and exercise 7 show that $2^n \geq n$ for all $n \in \mathbb{Z}_+$.
(c) Let Y^X denote the X-fold Cartesian product of Y with itself i.e. $\Pi_{x \in X} Y$. This consists of all ordered tuples, where the coordinates are indexed by elements of X and whose entries come from the set Y. Describe a bijection between Y^X and the set $Func(X, Y)$ of functions from X to Y.
(d) If $|Y| = n, |X| = m$ explain why $|Y^X| = n^m$.

9.
(a) Show that the set $\{0,1\}^\omega$ of infinite binary sequences is uncountable via a "Cantor diagonal argument".
(b) Fix $N \in \mathbb{Z}_+$. A sequence $(x_1, x_2, \ldots)$ is called N-periodic if $x_{i+N} = x_i$ for all $i \in \mathbb{Z}_+$. N is called the period. Explain why the set $S_N \subseteq \{0,1\}^\omega$ of N-periodic infinite binary sequences is finite and find its cardinality.
(c) Let $S = \cup_{N=1}^{\infty} S_N$ be the set of periodic infinite binary sequences (of any period). Show that S is a countably infinite set.
(d) Explain why there are uncountably many non-periodic infinite binary sequences.
(e) A sequence is called "eventually N-periodic" after m steps if $a_{i+N} = a_i$ for $i > m$. This allows the first m terms to be anything but then the rest of the sequence has to be N-periodic. Let $S_{m,N}$ be the set of infinite binary sequences which are eventually N-periodic after m steps. Then $S = \cup_{(m,N) \in \mathbb{Z}_+ \times \mathbb{Z}_+} S_{m,N}$ is the set of all "eventually periodic infinite binary sequences". Explain why this set is countable and why there are uncountably many infinite binary sequences which are not eventually periodic.

10.
A monic rational polynomial of degree n is a polynomial of the form $x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ where all the coefficients $c_i \in \mathbb{Q}$. Let S_n denote the set of all such polynomials.
(a) Give an explicit bijection $\mathbb{Q}^n \to S_n$ and use it to explain why S_n is countable.
(b) Let $S = \cup_{n=1}^{\infty} S_n$ be the set of all monic, non constant rational polynomials. Explain why this set is countable.
(c) For each nonzero polynomial p, it follows from basic algebra that the set of complex zeroes, $Z(p) = \{a \in \mathbb{C} | p(a) = 0\}$ is finite. Define a complex number to be **algebraic** if it is the zero of a nonzero, monic rational polynomial. Thus if we let A denote the set of algebraic numbers then $A = \cup_{p \in S} Z(p)$. Use this to explain why there are only countably many algebraic numbers.
(d) A complex number is called **transcendental** if it is not algebraic. Explain why there are uncountably many transcendental complex numbers.
(Comment: In (c), (d) one could work within the real numbers instead of the complex numbers with similar results.)

11. (Schroeder-Bernstein Theorem)
The purpose of this exercise is to prove a useful theorem called the Schroeder-Bernstein Theorem which says that if there exist injective maps $f : A \to B$ and $g : B \to A$ then there must exist a bijection between A and B. We will prove this in a specific way:
(a) We first will consider a "special case" where $C \subseteq A$ and there is an injective map $h : A \to C$. Using the principle of recursive definition we can define a sequence of sets $A_n, n \in \mathbb{Z}_+$ via $A_1 = A$ and $A_{n+1} = h(A_n)$ for $n \geq 1$. Similarly we can define a sequence of sets $C_n, n \in \mathbb{Z}_+$ via $C_1 = C$ and $C_{n+1} = h(C_n)$ for $n \geq 1$. Use the principle of mathematical induction to show that $A_1 \supseteq C_1 \supseteq A_2 \supseteq C_2 \supseteq A_3 \supseteq \ldots$ i.e., that $A_n \supseteq C_n \supseteq A_{n+1}$ for all $n \in \mathbb{Z}_+$. Also draw a Venn diagram of these nested sets - it should look like a set of concentric circles making "rings" - label things reasonably.
(b) Now we will define a bijection $\Psi : A \to C$ by $\Psi(x) = h(x)$ if $x \in A_n - C_n$ for some $n \in \mathbb{Z}_+$ and $\Psi(x) = x$ otherwise. Describe what Ψ does to the "rings" in your Venn diagram heuristically and use this to help you provide a proof that $\Psi : A \to C$ is a bijection.
(c) Prove the Schroeder-Bernstein Theorem:

Theorem 0.1 (Schroeder-Bernstein Theorem)**.** *Let A, B be sets, $f : A \to B, g : B \to A$ injective functions. Then there exists a bijection $\Theta : A \to B$.*

(Hint: Set $C = g(B) \subseteq A$ and $h : A \to C$ given by $h(a) = g(f(a))$ for all $a \in A$. Then explain why parts (a) and (b) apply and how they can be used to construct the bijection $\Theta : A \to B$.)
(d) Consider $A = (0, 1)$ the open unit interval and $B = (0, 1) \times (0, 1) \subseteq \mathbb{R}^2$. Construct explicit injective functions $f : A \to B$ and $g : B \to A$. (Hint: consider decimal expansions and "shuffling them"). Thus by the Schroeder-Bernstein theorem, there is a bijection between the open interval $(0, 1)$ and the "unit open square" $(0, 1) \times (0, 1)$ so they have the same cardinality. This shows that these two sets have the same "number" of "constituent atoms". However it is clear they don't have the "same shape" and later we will see indeed that there is no "shape preserving" bijection between them.