**MATH 240H: Homework 11: Assorted Topics and Applications.**
**Due Tuesday, April 23, 11:59PM on gradescope**

1. (a) Show that any set $X$ with the finite complement topology is compact directly from the definitions.
(b) Explain briefly why that $\mathbb{R}^1$ with the Zariski topology is compact.
(c) Let $\mathbb{Z}_\omega = \mathbb{Z}_+ \cup \{\omega\} = \{1, 2, 3, \dots, \omega\}$ be the ordered set obtained by adjoining a largest element $\omega$ to $\mathbb{Z}_+$. Show that $\mathbb{Z}_\omega$ is compact in its order topology.

2. Let $X$ be a topological space. Show that a finite union of compact subspaces of $X$ is compact.

3. Let $X$ be a Hausdorff topological space and $A, B$ compact subspaces with $A \cap B = \emptyset$. Show that there are **disjoint** sets $U, V$ which are **open in** $X$, such that $A \subseteq U, B \subseteq V$. [Hint: First prove it in the case $B$ is a single point $b$. Consider each $a \in A$ and find disjoint open sets $U_a, V_a$ such that $a \in U_a, b \in V_a$ and carefully construct $U$ and $V$ from these. When generalizing to the case where $B$ is not a single point, for each $b$ in $B$, first construct two disjoint open sets, one containing $A$ and one containing $b$.]

4. Let $(X, d)$ be a metric space. Given a bounded subset $A$, the diameter of $A$, denoted $diam(A)$ is defined by

$$diam(A) = sup\{d(x, y) | x, y \in A\}.$$

(a) Show that if $C$ is a compact subspace of $X$, then there exists $x, y \in C$ such that $d(x, y) = diam(C)$.
(b) Give an example of a bounded subset $A$ of $\mathbb{R}^2$ where there does not exist $x, y \in A$ such that $d(x, y) = diam(A)$.

5. Let $(G, \star)$ be a topological group. Let $A, B$ be subspaces of $G$.
(a) If $A$ is closed and $B$ is compact show that $A \star B$ is closed. (Hint: Let $c \in G, c \notin A \star B$. Let $f : G \times B \to G$ be given by $f(x, y) = x \star y^{-1}$ and show that $f^{-1}(G - A)$ is an open neighborhood of the slice $c \times B$. Use the tube lemma to find an open neighborhood $W$ of $c$ such that $W \times B \subseteq f^{-1}(G - A)$. Finish by explaning why your work shows $G - A \star B$ is open.)
(b) Let $A = \mathbb{Z}_+$ and $B = \{-n + \frac{1}{n} | n \geq 2, n \in \mathbb{Z}\}$. Show that $A$, $B$ are closed

subsets of $(\mathbb{R}, +)$ but $A + B = \{a + b | a \in A, b \in B\}$ is not closed. Thus (a) does not hold if $B$ is only required to be closed but not compact.

6. (a) Let $X$ be a topological space and $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$ be a nested sequence of nonempty closed subspaces. If $C_1$ is compact explain why $\cap_{n=1}^{\infty} C_n$ is nonempty. (Hint: Work within $C_1$ and consider $U_j = C_1 - C_j$ and explain why it is open in $C_1$.)
(b) Let $Y$ be a compact (nonempty) topological space and let $f : Y \to Y$ be continuous. Define recursively $Y_0 = Y$ and $Y_n = f(Y_{n-1})$ for $n \in \mathbb{Z}_+$. Show that $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots$ is a nested sequence of compact spaces. If $Y$ is assumed in addition to be Hausdorff explain why $Y_\infty = \cap_{n=1}^{\infty} Y_n$ is a nonempty compact subspace of $Y$ and $f : Y_\infty \to Y_\infty$.
(c) Show that $Y_\infty = \emptyset$ is possible if compactness is dropped in (b) by considering the example of $f : Y \to Y$, $Y = (0, 1)$, $f(t) = \frac{t}{2}$.
(d) Show that $f : Y_\infty \to Y_\infty$ is surjective when $Y$ is a compact, metric space. (Hint: Fix $b \in Y_\infty$ and first explain why there exist $a_k \in Y_k$ such that $f(a_k) = b$. Then explain why there is a subsequence of the $a_k$ that converge to some limit $\alpha \in Y_\infty$ and why $f(\alpha) = b$).

7. (a) Let $(Y, d)$ be a compact metric space and $f : Y \to Y$ be a **shrinking map** i.e., $d(f(a), f(b)) < d(a, b)$ for all $a \neq b \in Y$. Show that $f$ has a unique fixed point in $Y$, i.e., a unique $y_0 \in Y$ such that $f(y_0) = y_0$. (Hint: Use exercise 6 and then exercise 4 to show that $Y_\infty$ has diameter zero.)
(b) Give an example of a metric space $Y$ and a shrinking map $f : Y \to Y$ which does not have a fixed point.

8. (Consulting exercise 4 of homework 7 might be useful when doing this exercise.)
(a) Let $(X, d)$ be a metric space. Let $A \subseteq X$ be a nonempty subset. Recall $D_A(x) = inf\{d(x, a) | a \in A\}$ represents the distance of the point $x$ to the set $A$. Show that if $A$ is compact, this infimum is achieved i.e. given $x \in X$, there is a point $a_0 \in A$ which is closest to $x$.
(b) Recall the $\epsilon$-neighborhood of $A$, $A^\epsilon = \{x \in X | D_A(x) < \epsilon\}$ is an open set containing $A$. If $A$ is compact, show that if $V$ is an open set containing $A$, there exists $\epsilon > 0$ such that $A \subseteq A^\epsilon \subseteq V$.
(c) Give an example to show that (b) does not hold if $A$ is just assumed to be closed but is not compact.

9. Let $Mat_n(\mathbb{R})$ be topologized by identifying it with $\mathbb{R}^{n^2}$ with the standard topology as usual.
(a) Show that $Mat_n(\mathbb{Z})$ the set of $n \times n$ matrices with integer entries is a closed, discrete subspace of $Mat_n(\mathbb{R})$.
(b) Let $G$ be a compact subgroup of $GL_n(\mathbb{R})$. Show that there are only finitely many matrices in $G$ with all integer entries.
(c) A rational number can always be reduced to the form $\frac{m}{n}$ where $m$ and $n$ are relatively prime integers. Such a rational number is said to have denominator bounded by $N$ if $|n| \leq N$. Explain why there are only finitely many matrices in the orthogonal group $O(23)$ with rational entries whose denominator is bounded by $10^{20}$.

10. A topological space is said to be **Alexandrov** if arbitrary intersections of open sets are open.
(a) Show that if $X$ is an Alexandrov space, then for every $\alpha \in X$, there is a minimal open neighborhood $M(\alpha)$ of $\alpha$. (This means that for any other open neighborhood $U$ of $\alpha$, we have $M(\alpha) \subseteq U$).
(b) Explain why Alexandrov spaces are first countable.
(c) Explain why every finite topological space is Alexandrov.
(d) A pre-order $\leq$ on a set $X$ is just a reflexive and transitive relation. (Warning unlike an order we do not have anti-symmetry in general i.e. $\alpha \leq \beta$ and $\beta \leq \alpha$ does not imply $\alpha = \beta$ nor is the order complete i.e. given $\alpha, \beta$, they might be incomparable in the pre-order.) Show that if $X$ is an Alexandrov space, we can get a pre-order on $X$ by declaring $\alpha \leq \beta$ if and only if $M(\alpha) \subseteq M(\beta)$.
(e) Conversely, if $(Y, \leq)$ is a set with a pre-order, a "negative ray" is any subset $U$ with the property, $u \in U, y \in Y, y \leq u \implies y \in U$ (Thus a negative ray is any subset $U$ with the property that anything in the ambient set $Y$ which is less than an element of $U$ must also be in $U$). Show that the collection of negative rays is a topology on $Y$ which is Alexandrov and $M(\alpha) = \{y \in Y | y \leq \alpha\}$ is the minimal open set containing $\alpha$. (Warning this "pre-order Alexandrov" topology is not the same as the usual order topology when $\leq$ is an order).
(f) Show that the set $\{1, 2, 3, 4, 5, 6\}$ is pre-ordered by the "divides" relation i.e. $a \leq_1 b$ if an only if $\frac{b}{a} \in \mathbb{Z}$ (We denote this pre-ordering as $\leq_1$ as it is very different from the standard ordering $\leq$). Write down the open sets in the corresponding Alexandrov topology. Show that this space is path connected (for this last bit recall that you have checked that the two-point Sierpinski

space is path connected - use this and concatenate paths).