MATH 240H: Homework 10: Connectedness and Applications.
Due Saturday, April 13 at 11:59PM on gradescope

1. (a) Let X and Y be two topological spaces and $f : X \to Y$ be a homeomorphism between them. For any $x \in X$ explain why $X - \{x\}$ is (path) connected if and only if $Y - \{f(x)\}$ is (path) connected. Thus the sets

$$\{x \in X | X - \{x\} \text{ is (path) connected}\}$$

and

$$\{y \in Y | Y - \{y\} \text{ is (path) connected}\}$$

have the same cardinality.
(b) Let $a < b$ in $\mathbb{R}$ and give the intervals $[a, b], [a, b), (a, b)$ their standard subspace topology. Using ideas from part (a), show that no two of these intervals are homeomorphic. In other words show closed intervals, open intervals and half-open intervals have different homeomorphism types.
(c) Show that if A is a subspace of $\mathbb{R}$ with $|A| \geq 3$, then there exists a point $a \in A$ such that $A - \{a\}$ is not connected.
(d) Show that the circle S^1 does not embed in $\mathbb{R}^1$, i.e., show that S^1 is not homeomorphic to A where A is a subspace of $\mathbb{R}^1$.

2. Given a topological space X and a map $f : X \to X$, a fixed point of f is a point $p \in X$ such that $f(p) = p$. Show that any continuous map $f : [0, 1] \to [0, 1]$ has a fixed point. (Hint: Consider $g(x) = f(x) - x$ and the sign of $g(0), g(1)$.)

3. Let $p(x)$ be an odd degree real polynomial. Show that p has a root on the real line i.e., there exists $\alpha \in \mathbb{R}$ such that $p(\alpha) = 0$.

4. Let X be a path connected topological space and $f : X \to \mathbb{R}$ be a continuous function. Suppose for $a, b \in X$ we have the product $f(a)f(b) < 0$. Show that for any path $p : [0, 1] \to X$ between a and b, there exists a $c \in p([0, 1])$ such that $f(c) = 0$, i.e., there is a root of f somewhere on the image set of any path between a and b. (Note: paths are by definition always continuous.)

5. (a) Show that $O(n)$ is not connected for any $n \geq 1$.
(b) Show that $SO(2)$ is path connected.

6. (a) Let $n \geq 1$ and $T : S^n \to \mathbb{R}$ be a continuous map. Show that there exist antipodal points $u, -u \in S^n$ such that $T(u) = T(-u)$. (Hint: Suppose not and consider the function $g(x) = \frac{T(x)-T(-x)}{|T(x)-T(-x)|}$. Show $g(-x) = -g(x)$ and $g : S^n \to S^0$ is onto and proceed from there.) (Some people like to use this to say that if at any given time, assuming the temperature profile on the surface of the earth is continuous, then there must be two places on opposite sides of the earth which have exactly the same temperature.)
(b) Let $\hat{u} \in S^1$, describe the set $H_{\hat{u}} = \{\hat{x} \in \mathbb{R}^2 | \hat{x} \cdot \hat{u} \geq 0\}$.
(c) Let A be a measurable subset of $\mathbb{R}^2$ (this just means it has a well-defined area), then one can show that the function $M : S^1 \to \mathbb{R}$ given by $M(\hat{u}) = Area(A \cap H_{\hat{u}})$ is a continuous function of $\hat{u}$. You may use this freely in this exercise. Explain why your work in (a),(b) shows that there exists a line through the origin which bisects A into two regions of equal area, i.e., the area of the part of A on and to one side of the line is the same as the area of the part of A on and to the other side of the line.

7. (a) Show that if $\{X_\alpha\}_{\alpha \in I}$ is a collection of path-connected topological spaces. Then the product $\times_{\alpha \in I} X_\alpha$ (with the product topology) is also a path-connected space.
(b) $A \subset \mathbb{R}^\omega$ is a convex set if whenever $a_1, a_2 \in A$ then the line segment between them, $\{(1-t)a_1 + ta_2 | 0 \leq t \leq 1\}$ is contained in A as well. Show that any convex subset of $\mathbb{R}^\omega$ is path-connected when endowed with the subspace topology coming from the product topology on $\mathbb{R}^\omega$.
(c) In contrast show that $(\mathbb{R}^\omega, \tau_{box})$ is not even connected. (Hint: Let $U = \{$ bounded sequences $\}$ and $V = \{$ unbounded sequences $\}$ and show that U, V are open in the box topology.)

8. Let A be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 - A$ is path connected. (Hint: How many lines are there passing through a given point in $\mathbb{R}^2$?) (Note in particular you will have proven that $\mathbb{R}^2 - \mathbb{Q}^2$ is path connected.)

9. In class we showed that $GL_n(\mathbb{R})$ is not connected. In this exercise you will show that its complex variant is in fact path connected. Let

$$GL_n(\mathbb{C}) = \{\mathbb{A} \in Mat_n(\mathbb{C}) | det(\mathbb{A}) \neq 0\}$$

be the group of invertible $n \times n$ complex matrices.
(a) Show that $GL_1(\mathbb{C}) = \mathbb{C} - \{0\}$ is path connected.

(b) Let $\mathbb{A} \in GL_n(\mathbb{C})$, then A can be put into Jordan canonical form:

$$\mathbb{A} = \mathbb{S} \begin{bmatrix} \lambda_1 & t_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & t_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \mathbb{S}^{-1}$$

Here in the middle matrix, $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathbb{A}$. Basically this says that any complex matrix can be "nearly diagonalized" except for some entries t_i right above the main diagonal.
Check that we can define a path $p : [0,1] \to GL_n(\mathbb{C})$ via the formula:

$$p(x) = \mathbb{S} \begin{bmatrix} \lambda_1 & (1-x)t_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & (1-x)t_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \mathbb{S}^{-1}$$

(You should check that p is continuous and that the outputs of p do lie in $GL_n(\mathbb{C})$). Furthermore check that $p(0) = \mathbb{A}$ and $p(1) = \mathbb{S}\mathbb{D}\mathbb{S}^{-1}$ where $\mathbb{D}$ is diagonal with entries the eigenvalues of $\mathbb{A}$.
(c) Using the path connectedness of $\mathbb{C} - \{0\}$ show how to define a path $q : [0,1] \to GL_n(\mathbb{C})$ such that $q(0) = \mathbb{S}\mathbb{D}\mathbb{S}^{-1}$ and $q(1) = \mathbb{S}\mathbb{I}\mathbb{S}^{-1} = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix.
(d) Explain why there is a path in $GL_n(\mathbb{C})$ from any $\mathbb{A} \in GL_n(\mathbb{C})$ to $\mathbb{I}$. Use this to explain why $GL_n(\mathbb{C})$ is path connected.

10. Let X denote the rational points of the interval $[0,1] \times \{0\}$ in $\mathbb{R}^2$. Let T denote the union of all line segments joining the point $p = 0 \times 1$ to points of X.
(a) Show that T is path connected but is locally connected only at the point p.
(b) Find a subspace of $\mathbb{R}^2$ which is path connected but is locally connected at none of its points.

11. Show that the two point space $X = \{a, b\}$ with topology $\tau = \{\emptyset, X, \{a\}\}$ is path connected. (Hint: Explicitly define a path $p : [0,1] \to \{a, b\}$ with $p(0) = a, p(1) = b$ and show it is continuous!)