MATH 240H: Homework 1: Logic, sets and functions.
Due on Gradescope, Saturday, Jan 27, 11:59PM

1. A double implication $p \leftrightarrow q$ is read "p if and only if q" and often abbreviated "p iff q". $p \leftrightarrow q$ is defined so that it is logically equivalent to $(p \to q) \wedge (q \to p)$. Thus in general to show $p \leftrightarrow q$ we often break the proof up into two steps, one showing $p \to q$ and the other showing $q \to p$ (unless it is reasonably easy to show both at once with the same argument.) In each of the following double implications, determine if the double implication is true for all sets A, B, C or not. If it is false, determine if the implication holds in any direction (left to right or right to left) at least. **No proofs required** but you might want to consider examples to help you decide things. For each part make sure to state if the double implication works or if not, which implications (left to right or right to left) do work.
(a) $(A \subseteq B) \wedge (A \subseteq C) \leftrightarrow A \subseteq (B \cup C)$.
(b) $(A \subseteq B) \vee (A \subseteq C) \leftrightarrow A \subseteq (B \cup C)$.
(c) $(A \subseteq B) \wedge (A \subseteq C) \leftrightarrow A \subseteq (B \cap C)$.
(d) $(A \subseteq B) \vee (A \subseteq C) \leftrightarrow A \subseteq (B \cap C)$.

2. Recall that $A \subseteq B$ holds if for all x, $(x \in A) \to (x \in B)$. Two sets A and B are equal if and only if they have the same elements. Thus $A = B$ holds if and only if $(x \in A) \leftrightarrow (x \in B)$ is true for all x. This in turn is equivalent to showing the implication $(x \in A) \to (x \in B)$ and the implication $(x \in B) \to (x \in A)$. These in turn are equivalent to showing $A \subseteq B$ and $B \subseteq A$. Thus in general, to show two sets A and B have $A = B$ it is equivalent to show $(A \subseteq B) \wedge (B \subseteq A)$. Determine if the following equalities of sets are true for all sets A, B, C. If they are not true, indicate if one side of the set equality is a subset of the other side at least. **No proofs are needed** - just state your answer - considering examples might be useful to you in deciding the matter.
(a) $A - (A - B) = B$.
(b) $A - (B - A) = A - B$.
(c) $A \cap (B - C) = (A \cap B) - (A \cap C)$.
(d) $A \cup (B - C) = (A \cup B) - (A \cup C)$.

3. Determine which of the following set identities involving Cartesian products is true for all sets A, B, C, D. If an identity is false, indicate if one side of the set equality is at least a subset of the other side. Again **no proofs**

are necessary in this question, just state your answer. It might be useful to draw "box"-pictures for the Cartesian products (as if they were cartesian products of real intervals) to help decide.
(a) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.
(b) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(c) $A \times (B - C) = (A \times B) - (A \times C)$.
(d) $(A - B) \times (C - D) = (A \times C - B \times C) - (A \times D)$.
(e) $(A \times B) - (C \times D) = (A - C) \times (B - D)$.

4. Let $D = \{(x, y)|x^2 + y^2 \leq 1\}$ be the unit disk of radius 1 about the origin in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.
(a) Show that if $D \subseteq A \times B$ where A, B are subsets of $\mathbb{R}$ then $[-1, 1] \subseteq (A \cap B)$.
(b) Show that D is a **proper** subset of $[-1, 1] \times [-1, 1]$.
(Note: By (a), the "smallest" Cartesian product of two subsets of the real numbers that contains D is $[-1, 1] \times [-1, 1]$ but by (b), $[-1, 1] \times [-1, 1]$ is not D. Thus D is an example of a subset of the plane which is NOT a cartesian product of two subsets of the real numbers.)

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f(x, y) = x^2 + y^2$.
(a) Draw a picture of the preimage set $f^{-1}(\{1\})$ as a subset of $\mathbb{R}^2$. Describe this set geometrically.
(b) Draw a picture of the preimage set $f^{-1}([1, 2))$. In your picture use dotted lines for a boundary if it is not in the set and solid lines if the boundary is in the set.
(c) Let $D = \{(x, y) \in \mathbb{R}^2|x^2 + y^2 \leq 1\}$. Describe $f(D)$ as a subset of the real line using interval notation.

6. Let $f : A \to B$ be a function. Let B_0, B_1 be subsets of B. Prove the following:
(a) If $B_0 \subseteq B_1$ then $f^{-1}(B_0) \subseteq f^{-1}(B_1)$.
(b) Prove that $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.
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To help give you ideas for your proof, here is a proof of the related fact that $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.
Proof:

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\begin{aligned}
x \in f^{-1}(B_0 \cap B_1) &\leftrightarrow f(x) \in (B_0 \cap B_1)\\
&\leftrightarrow (f(x) \in B_0) \wedge (f(x) \in B_1)\\
&\leftrightarrow (x \in f^{-1}(B_0)) \wedge (x \in f^{-1}(B_1))\\
&\leftrightarrow x \in (f^{-1}(B_0) \cap f^{-1}(B_1)).
\end{aligned}
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Thus the sets $f^{-1}(B_0 \cap B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1)$ have exactly the same elements and so are equal sets.

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(c) Prove that $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.
(Note: the exercise above shows that the process of taking preimages preserves inclusions, unions, intersections and differences of sets.)

7. Let $f : A \to B$ be a function. Let A_0, A_1 be subsets of A. Prove the following:
(a) $A_0 \subseteq A_1 \to f(A_0) \subseteq f(A_1)$.
(b) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.
(c) $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$. Also give an example to show that set equality $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$ does not hold in general.
(d) $f(A_0 - A_1) \supseteq f(A_0) - f(A_1)$. Also give an example to show that set equality $f(A_0 - A_1) = f(A_0) - f(A_1)$ does not hold in general.
(Note the exercise above shows that the process of taking images preserves inclusions and unions but doesn't preserve intersections and differences in general.)

8. Let $n \geq 1$ and let $Mat_n(\mathbb{R})$ denote the set of $n \times n$ matrices with real entries. If $A \in Mat_n(\mathbb{R})$ recall A^T, the transpose of A, is the matrix whose columns are the rows of A, i.e., the (i, j)-entry of A^T is the (j, i)-entry of A. Let I denote the $n \times n$ identity matrix. Let us define $O(n) = \{A \in Mat_n(\mathbb{R}) | AA^T = I\}$. This is the set of "$n \times n$ orthogonal matrices".
(a) Recall that if $C = AB$ is a product of matrices then the (i, j)-entry of C, c_{ij} is given by the dot product of the ith row of A with the jth column of B. Use this to give an explanation of why a matrix $A \in Mat_n(\mathbb{R})$ is in $O(n)$ if and only if its rows are an orthonormal set of vectors i.e., each row has unit Euclidean length and any two distinct rows are orthogonal to each other, i.e., have dot product zero.

(b) Show that any matrix of the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, $\theta \in \mathbb{R}$ is in $O(2)$.
(c) Show that $A \in O(n) \to det(A) = \pm 1$. [Hint: Start with the equation $AA^T = I$.]
(d) Within the set $Mat_2(\mathbb{R})$, show the converse implication to (c) is false in general by giving a counterexample.