Exercises

1. Show that $\mathbb{Q}$ is countably infinite.

2. Show that the maps f and g of Examples 1 and 2 are bijections.

3. Let X be the two-element set $\{0, 1\}$. Show there is a bijective correspondence between the set $\mathcal{P}(\mathbb{Z}_+)$ and the cartesian product X^ω.

4. (a) A real number x is said to be ***algebraic*** (over the rationals) if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients a_i. Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable.

(b) A real number is said to be ***transcendental*** if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable. (It is a somewhat surprising fact that only two transcendental numbers are familiar to us: e and π. Even proving these two numbers transcendental is highly nontrivial.)

5. Determine, for each of the following sets, whether or not it is countable. Justify your answers.

(a) The set A of all functions $f : \{0, 1\} \to \mathbb{Z}_+$.

(b) The set B_n of all functions $f : \{1, \ldots, n\} \to \mathbb{Z}_+$.

(c) The set $C = \bigcup_{n \in \mathbb{Z}_+} B_n$.

(d) The set D of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.

(e) The set E of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$.

(f) The set F of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ that are "eventually zero." [We say that f is ***eventually zero*** if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$.]

(g) The set G of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually 1.

(h) The set H of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually constant.

(i) The set I of all two-element subsets of $\mathbb{Z}_+$.

(j) The set J of all finite subsets of $\mathbb{Z}_+$.

6. We say that two sets A and B ***have the same cardinality*** if there is a bijection of A with B.

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \longrightarrow B,$$

then A and B have the same cardinality. [*Hint:* Define $A_1 = A$, $B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$. Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$

(b) *Theorem (Schroeder-Bernstein theorem). If there are injections $f : A \to C$ and $g : C \to A$, then A and C have the same cardinality.*

7. Show that the sets D and E of Exercise 5 have the same cardinality.

8. Let X denote the two-element set $\{0, 1\}$; let $\mathcal{B}$ be the set of *countable* subsets of X^ω. Show that X^ω and $\mathcal{B}$ have the same cardinality.

9. (a) The formula

$$(*) \qquad \begin{aligned} h(1) &= 1, \\ h(2) &= 2, \\ h(n) &= [h(n+1)]^2 - [h(n-1)]^2 \quad \text{for } n \geq 2 \end{aligned}$$

is not one to which the principle of recursive definition applies. Show that nevertheless there does exist a function $h : \mathbb{Z}_+ \to \mathbb{R}$ satisfying this formula. [*Hint:* Reformulate $(*)$ so that the principle will apply and require h to be positive.]

(b) Show that the formula $(*)$ of part (a) does not determine h uniquely. [*Hint:* If h is a positive function satisfying $(*)$, let $f(i) = h(i)$ for $i \neq 3$, and let $f(3) = -h(3)$.]

(c) Show that there is no function $h : \mathbb{Z}_+ \to \mathbb{R}$ satisfying the formula

$$\begin{aligned} h(1) &= 1, \\ h(2) &= 2, \\ h(n) &= [h(n+1)]^2 + [h(n-1)]^2 \quad \text{for } n \geq 2. \end{aligned}$$

*§8 The Principle of Recursive Definition

Before considering the general form of the principle of recursive definition, let us first prove it in a specific case, that of Lemma 7.2. That should make the underlying idea of the proof much clearer when we consider the general case.

So, given the infinite subset C of $\mathbb{Z}_+$, let us consider the following recursion formula for a function $h : \mathbb{Z}_+ \to C$:

$$(*) \qquad \begin{aligned} h(1) &= \text{smallest element of } C, \\ h(i) &= \text{smallest element of } [C - h(\{1, \ldots, i-1\})] \quad \text{for } i > 1. \end{aligned}$$

We shall prove that there exists a unique function $h : \mathbb{Z}_+ \to C$ satisfying this recursion formula.

The first step is to prove that there exist functions defined on *sections* $\{1, \ldots, n\}$ of $\mathbb{Z}_+$ that satisfy $(*)$:

Lemma 8.1. *Given $n \in \mathbb{Z}_+$, there exists a function*

$$f : \{1, \ldots, n\} \to C$$

that satisfies $(*)$ for all i in its domain.