is not empty.]

12. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$. (Such a map is called a ***perfect map***.) Show that if Y is compact, then X is compact. [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, there is a neighborhood W of y such that $p^{-1}(W)$ is contained in U.]

13. Let G be a topological group.

(a) Let A and B be subspaces of G. If A is closed and B is compact, show $A \cdot B$ is closed. [*Hint:* If c is not in $A \cdot B$, find a neighborhood W of c such that $W \cdot B^{-1}$ is disjoint from A.]

(b) Let H be a subgroup of G; let $p : G \to G/H$ be the quotient map. If H is compact, show that p is a closed map.

(c) Let H be a compact subgroup of G. Show that if G/H is compact, then G is compact.

§27 Compact Subspaces of the Real Line

The theorems of the preceding section enable us to construct new compact spaces from existing ones, but in order to get very far we have to find some compact spaces to start with. The natural place to begin is the real line; we shall prove that every closed interval in $\mathbb{R}$ is compact. Applications include the extreme value theorem and the uniform continuity theorem of calculus, suitably generalized. We also give a characterization of all compact subspaces of $\mathbb{R}^n$, and a proof of the uncountability of the set of real numbers.

It turns out that in order to prove every closed interval in $\mathbb{R}$ is compact, we need only *one* of the order properties of the real line—the least upper bound property. We shall prove the theorem using only this hypothesis; then it will apply not only to the real line, but to well-ordered sets and other ordered sets as well.

Theorem 27.1. *Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.*

Proof. *Step 1.* Given $a < b$, let $\mathcal{A}$ be a covering of $[a, b]$ by sets open in $[a, b]$ in the subspace topology (which is the same as the order topology). We wish to prove the existence of a finite subcollection of $\mathcal{A}$ covering $[a, b]$. First we prove the following: If x is a point of $[a, b]$ different from b, then there is a point $y > x$ of $[a, b]$ such that the interval $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

If x has an immediate successor in X, let y be this immediate successor. Then $[x, y]$ consists of the two points x and y, so that it can be covered by at most two elements of $\mathcal{A}$. If x has no immediate successor in X, choose an element A of $\mathcal{A}$ containing x. Because $x \neq b$ and A is open, A contains an interval of the form $[x, c)$, for some c in $[a, b]$. Choose a point y in (x, c); then the interval $[x, y]$ is covered by the single element A of $\mathcal{A}$.

Step 2. Let C be the set of all points $y > a$ of $[a, b]$ such that the interval $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$. Applying Step 1 to the case $x = a$, we see that there exists at least one such y, so C is not empty. Let c be the least upper bound of the set C; then $a < c \leq b$.

Step 3. We show that c belongs to C; that is, we show that the interval $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Choose an element A of $\mathcal{A}$ containing c; since A is open, it contains an interval of the form $(d, c]$ for some d in $[a, b]$. If c is not in C, there must be a point z of C lying in the interval (d, c), because otherwise d would be a smaller upper bound on C than c. See Figure 27.1. Since z is in C, the interval $[a, z]$ can be covered by finitely many, say n, elements of $\mathcal{A}$. Now $[z, c]$ lies in the single element A of $\mathcal{A}$, hence $[a, c] = [a, z] \cup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$. Thus c is in C, contrary to assumption.

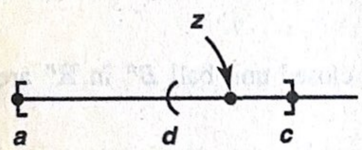

Figure 27.1

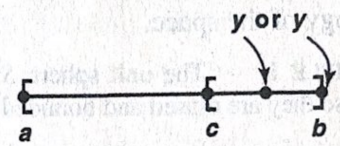

Figure 27.2

Step 4. Finally, we show that $c = b$, and our theorem is proved. Suppose that $c < b$. Applying Step 1 to the case $x = c$, we conclude that there exists a point $y > c$ of $[a, b]$ such that the interval $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$. See Figure 27.2. We proved in Step 3 that c is in C, so $[a, c]$ can be covered by finitely many elements of $\mathcal{A}$. Therefore, the interval

$$[a, y] = [a, c] \cup [c, y]$$

can also be covered by finitely many elements of $\mathcal{A}$. This means that y is in C, contradicting the fact that c is an upper bound on C. ■

Corollary 27.2. *Every closed interval in $\mathbb{R}$ is compact.*

Now we characterize the compact subspaces of $\mathbb{R}^n$:

Theorem 27.3. *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric ρ.*

Proof. It will suffice to consider only the metric ρ; the inequalities

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

imply that A is bounded under d if and only if it is bounded under ρ.

Suppose that A is compact. Then, by Theorem 26.3, it is closed. Consider the collection of open sets

$$\{B_\rho(\mathbf{0}, m) \mid m \in \mathbb{Z}_+\},$$

whose union is all of $\mathbb{R}^n$. Some finite subcollection covers A. It follows that $A \subset B_\rho(\mathbf{0}, M)$ for some M. Therefore, for any two points x and y of A, we have $\rho(x, y) \leq 2M$. Thus A is bounded under ρ.

Conversely, suppose that A is closed and bounded under ρ; suppose that $\rho(x, y) \leq N$ for every pair x, y of points of A. Choose a point x_0 of A, and let $\rho(x_0, \mathbf{0}) = b$. The triangle inequality implies that $\rho(x, \mathbf{0}) \leq N + b$ for every x in A. If $P = N + b$, then A is a subset of the cube $[-P, P]^n$, which is compact. Being closed, A is also compact. ∎

Students often remember this theorem as stating that the collection of compact sets in a *metric space* equals the collection of closed and bounded sets. This statement is clearly ridiculous as it stands, because the question as to which sets are bounded depends for its answer on the metric, whereas which sets are compact depends only on the topology of the space.

EXAMPLE 1. The unit sphere S^{n-1} and the closed unit ball B^n in $\mathbb{R}^n$ are compact because they are closed and bounded. The set

$$A = \{x \times (1/x) \mid 0 < x \leq 1\}$$

is closed in $\mathbb{R}^2$, but it is not compact because it is not bounded. The set

$$S = \{x \times (\sin(1/x)) \mid 0 < x \leq 1\}$$

is bounded in $\mathbb{R}^2$, but it is not compact because it is not closed.

Now we prove the extreme value theorem of calculus, in suitably generalized form.

Theorem 27.4 (Extreme value theorem). *Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

The extreme value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact. We show that A has a largest element M and a smallest element m. Then since m and M belong to A, we must have $m = f(c)$ and $M = f(d)$ for some points c and d of X.

If A has no largest element, then the collection

$$\{(-\infty, a) \mid a \in A\}$$

forms an open covering of A. Since A is compact, some finite subcollection

$$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$

covers A. If a_i is the largest of the elements $a_1, \ldots a_n$, then a_i belongs to none of these sets, contrary to the fact that they cover A.

A similar argument shows that A has a smallest element. ∎

Now we prove the uniform continuity theorem of calculus. In the process, we are led to introduce a new notion that will prove to be surprisingly useful, that of a *Lebesgue number* for an open covering of a metric space. First, a preliminary notion:

Definition. Let (X, d) be a metric space; let A be a nonempty subset of X. For each $x \in X$, we define the ***distance from x to A*** by the equation

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

It is easy to show that for fixed A, the function $d(x, A)$ is a continuous function of x: Given $x, y \in X$, one has the inequalities

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a),$$

for each $a \in A$. It follows that

$$d(x, A) - d(x, y) \leq \inf d(y, a) = d(y, A),$$

so that

$$d(x, A) - d(y, A) \leq d(x, y).$$

The same inequality holds with x and y interchanged; continuity of the function $d(x, A)$ follows.

Now we introduce the notion of Lebesgue number. Recall that the *diameter* of a bounded subset A of a metric space (X, d) is the number

$$\sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Lemma 27.5 (The Lebesgue number lemma). *Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it.*

The number δ is called a ***Lebesgue number*** for the covering $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be an open covering of X. If X itself is an element of $\mathcal{A}$, then any positive number is a Lebesgue number for $\mathcal{A}$. So assume X is not an element of $\mathcal{A}$.

Choose a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X - A_i$, and define $f : X \to \mathbb{R}$ by letting $f(x)$ be the average of the numbers $d(x, C_i)$. That is,

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i).$$

We show that $f(x) > 0$ for all x. Given $x \in X$, choose i so that $x \in A_i$. Then choose ϵ so the ϵ-neighborhood of x lies in A_i. Then $d(x, C_i) \geq \epsilon$, so that $f(x) \geq \epsilon/n$.

Since f is continuous, it has a minimum value δ; we show that δ is our required Lebesgue number. Let B be a subset of X of diameter less than δ. Choose a point x_0 of B; then B lies in the δ-neighborhood of x_0. Now

$$\delta \leq f(x_0) \leq d(x_0, C_m),$$

where $d(x_0, C_m)$ is the largest of the numbers $d(x_0, C_i)$. Then the δ-neighborhood of x_0 is contained in the element $A_m = X - C_m$ of the covering $\mathcal{A}$. ∎

Definition. A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be ***uniformly continuous*** if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \Longrightarrow d_Y(f(x_0), f(x_1)) < \epsilon.$$

Theorem 27.6 (Uniform continuity theorem). *Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.*

Proof. Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$. Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls under f. Choose δ to be a Lebesgue number for the covering $\mathcal{A}$. Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has diameter less than δ, so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \epsilon$, as desired. ∎

Finally, we prove that the real numbers are uncountable. The interesting thing about this proof is that it involves no algebra at all—no decimal or binary expansions of real numbers or the like—just the order properties of $\mathbb{R}$.

Definition. If X is a space, a point x of X is said to be an ***isolated point*** of X if the one-point set $\{x\}$ is open in X.

Theorem 27.7. *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. *Step 1.* We show first that given any nonempty open set U of X and any point x of X, there exists a nonempty open set V contained in U such that $x \notin \bar{V}$.

Choose a point y of U different from x; this is possible if x is in U because x is not an isolated point of X and it is possible if x is not in U simply because U is nonempty. Now choose disjoint open sets W_1 and W_2 about x and y, respectively. Then the set $V = W_2 \cap U$ is the desired open set; it is contained in U, it is nonempty because it contains y, and its closure does not contain x. See Figure 27.3.

Step 2. We show that given $f : \mathbb{Z}_+ \to X$, the function f is not surjective. It follows that X is uncountable.

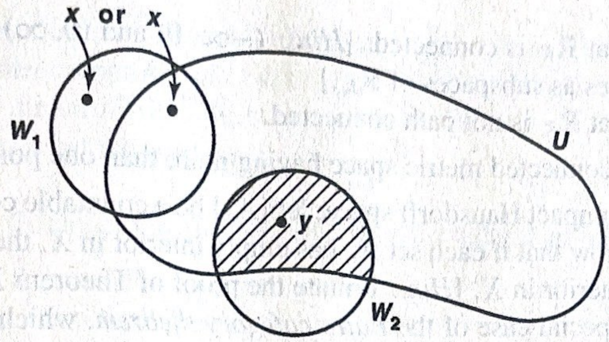

Figure 27.3

Let $x_n = f(n)$. Apply Step 1 to the nonempty open set $U = X$ to choose a nonempty open set $V_1 \subset X$ such that $\bar{V}_1$ does not contain x_1. In general, given V_{n-1} open and nonempty, choose V_n to be a nonempty open set such that $V_n \subset V_{n-1}$ and $\bar{V}_n$ does not contain x_n. Consider the nested sequence

$$\bar{V}_1 \supset \bar{V}_2 \supset \cdots$$

of nonempty closed sets of X. Because X is compact, there is a point $x \in \bigcap \bar{V}_n$, by Theorem 26.9. Now x cannot equal x_n for any n, since x belongs to $\bar{V}_n$ and x_n does not. ■

Corollary 27.8. *Every closed interval in* $\mathbb{R}$ *is uncountable.*

Exercises

1. Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bound property.
2. Let X be a metric space with metric d; let $A \subset X$ be nonempty.
   (a) Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.
   (b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.
   (c) Define the *ϵ-neighborhood* of A in X to be the set

   $$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$

   Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.
   (d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.
   (e) Show the result in (d) need not hold if A is closed but not compact.
3. Recall that $\mathbb{R}_K$ denotes $\mathbb{R}$ in the K-topology.
   (a) Show that $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.

(b) Show that $\mathbb{R}_K$ is connected. [*Hint:* $(-\infty, 0)$ and $(0, \infty)$ inherit their usual topologies as subspaces of $\mathbb{R}_K$.]
(c) Show that $\mathbb{R}_K$ is not path connected.

4. Show that a connected metric space having more than one point is uncountable.

5. Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X. [*Hint:* Imitate the proof of Theorem 27.7.]

This is a special case of the *Baire category theorem*, which we shall study in Chapter 8.

6. Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, define A_n by the equation

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

The intersection

$$C = \bigcap_{n \in \mathbb{Z}_+} A_n$$

is called the ***Cantor set***; it is a subspace of $[0, 1]$.
(a) Show that C is totally disconnected.
(b) Show that C is compact.
(c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$; and show that the end points of these intervals lie in C.
(d) Show that C has no isolated points.
(e) Conclude that C is uncountable.

§28 Limit Point Compactness

As indicated when we first mentioned compact sets, there are other formulations of the notion of compactness that are frequently useful. In this section we introduce one of them. Weaker in general than compactness, it coincides with compactness for metrizable spaces.

Definition. A space X is said to be ***limit point compact*** if every infinite subset of X has a limit point.

In some ways this property is more natural and intuitive than that of compactness. In the early days of topology, it was given the name "compactness," while the open covering formulation was called "bicompactness." Later, the word "compact" was shifted to apply to the open covering definition, leaving this one to search for a new