# MTH 236, Spring 2024 - Homework 1

## Due on Friday, January 26 at 11:59pm on gradescope

*The following standard conventions are used:* $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ is the set of integers; $\mathbb{N} = \mathbb{Z}^+ = \{1, 2, \ldots\}$ is the set of natural numbers, $\mathbb{R}$ is the set of real numbers.

**Question 1.** Show that the given intervals have the same cardinality by giving a formula for a one-to-one and onto function $f$ (a bijection) from the first to the second.

(a) $[0, 1]$ and $[1, 5]$.

(b) $(2, 4)$ and $(6, 26)$.

(c) $(a, b]$ and $(c, d]$ (assume $a < b$ and $c < d$)

**Question 2.** Prove that $(0, 1)$ has the same cardinality as $\mathbb{R}$. [Hint: try to find a function from calculus that maps $\mathbb{R}$ to an open interval bijectively, and then translate and scale this function appropriately to make the range equal to $(0, 1)$].

**Question 3.** Let $A = \{a, b, c, d\}$ with $a, b, c$, and $d$ distinct. Explain your answers to the following:

(a) How many subsets are there of $A$?

(b) How many partitions are there of $A$?

(c) How many distinct functions $\phi : A \to A$ exist?

(d) How many distinct injective functions $\phi : A \to A$ exist?

(e) How many distinct relations can be put on $A$?

(f) How many distinct equivalence relations can be put on $A$?

**Question 4.** For each of the following you are given a set and relation. Determine (with reasoning) whether it is an equivalence relation on this given set. If it is, describe the partition arising from it. If it is not, list all of the properties of equivalence relations that it fails.

(a) $n\mathcal{R}m$ in $\mathbb{Z}$ if $n < m$.

(b) $z\mathcal{R}w$ in $\mathbb{R}$ if $zw > 0$.

(c) $z\mathcal{R}w$ in $\mathbb{R}$ if $|z| = |w|$.

(d) $z\mathcal{R}w$ in $\mathbb{R}$ if $|z-w| \geq 1$.

(e) Let FS($\mathbb{N}$) be the set of all finite subsets of $\mathbb{N}$. $A\mathcal{R}B$ in FS($\mathbb{N}$) if $|A| = |B|$.

(f) Let $\{0,1\}^{\mathbb{N}}$ be the set of all functions from $\mathbb{N}$ to $\{0,1\}$. $f\mathcal{R}g$ in $\{0,1\}^{\mathbb{N}}$ if $f(1) = g(1)$.

**Question 5.** Fix an integer $n \in \mathbb{Z}$ and define a relation $\sim$ on $\mathbb{Z}$ by

$$r \sim s \iff r - s = nq \text{ for some } q \in \mathbb{Z},$$

i.e., iff $r - s$ is divisible by $n$.

(a) Prove that $\sim$ is an equivalence relation. (It is called the "modulo $n$" equivalence relation. Notation: If $a \sim b$, we say that "$a$ is congruent to $b$ mod $n$" and write $a \equiv b \pmod{n}$. )

(b) Describe the equivalence classes for $n = 4$.

**Question 6.** Give an example of a set $S$ and a relation on the set $S$ that is

(a) reflexive and symmetric, but not transitive;

(b) reflexive and transitive, but not symmetric;

(c) symmetric and transitive, but not reflexive.