MTH 236, Spring 2024 - Homework 9

Due on Friday, March 29 at 11:59pm on gradescope

1. True or false? Provide brief justifications for your answers. Throughout, G is a group, H is a subgroup of G and N is a normal subgroup of G.
   (a) There is a natural way to turn the set of cosets of H in G into a group.
   (b) If G is abelian, then there is a natural way to turn the set of cosets of H in G into a group.
   (c) If G/N is infinite then so is G.
   (d) If G/N is nonabelian then so is G.
   (e) If G is cyclic then so is G/N.
   (f) If N and G/N are both abelian then so is G.
   (g) Because N is normal, $gNg^{-1} = N$ for all $g \in G$.
   (h) If H is the only subgroup of G of order $d < \infty$, then H must be a normal subgroup of G.
   (i) Let G be a group and let $H, K \leq G$ be normal subgroups of G. Must $H \cap K$ be a normal subgroup of G? If so, prove it, and if not, give a counterexample.

2. Find the order of each of the following factor groups. Briefly justify your answers.
   (a) $(\mathbb{Z}_9 \times \mathbb{Z}_{35})/(\langle 3 \rangle \times \langle 25 \rangle)$.
   (b) $(\mathbb{Z}_9 \times \mathbb{Z}_{35})/(\{0\} \times \langle 11 \rangle)$.
   (c) $(\mathbb{Z}_{19} \times \mathbb{Z}_{24})/\langle (1, 1) \rangle$.

3. Classify the given group according to the fundamental theorem of finitely generated abelian groups. Briefly justify your answers.
   (a) $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle (0, 2) \rangle$.
   (b) $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_4)/\langle (3, 0, 0) \rangle$.
   (c) $(\mathbb{Z} \times \mathbb{Z})/\langle (2, 2) \rangle$.
   (d) $(\mathbb{Z} \times \mathbb{Z})/\langle (1, 2) \rangle$.

4. Find the order of each of the following elements in the factor groups. Briefly justify your answers.

   (a) $10 + \langle 8 \rangle$ in $\mathbb{Z}_{24}/\langle 8 \rangle$.

   (b) $(1,7) + \langle (1,1) \rangle$ in $(\mathbb{Z}_6 \times \mathbb{Z}_9)/\langle (1,1) \rangle$.

   (c) $(2,3) + \langle (1,2) \rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (1,2) \rangle$.

   (d) $(3,2) + \langle (1,2) \rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (1,2) \rangle$.

If X is a G-set, for $g \in G$ and $x \in X$ we will denote the action of g on x by gx, or by $g * x$ if it would cause confusion to write gx.

5. Let $G = S_3$ act on the set $X = \{1, 2, 3\}$ in the usual way (e.g. $(1\ 2)2 = 1$, and $(1\ 2)3 = 3$, etc).

   (a) For each $i \in \{1, 2, 3\}$, find the isotropy subgroup G_i.

   (b) For each $g \in S_3$, find X_g.

   (c) Show that the action is *transitive*, that is, it has exactly one orbit.

6. Let $G = D_4 = \langle \rho, \tau \rangle$. Let X be the set of subgroups of G (note that $|X| = 10$). Let D_4 act on X by conjugation, so that $g * H = i_g(H) = gHg^{-1}$.

   (a) Compute X_ρ.

   (b) What are the orbits of this action?

   (c) Using your answer to part (b), determine all the normal subgroups of D_4.

7. Consider squares where the edges are painted different colors. Call two squares *indistinguishable* if there is a way of physically moving the first square so that the colors of its edges match exactly with those of the second square. For example, if in square A, the top and bottom edges are blue and the left and right edges are red, and in square B the left and right edges are blue and top and bottom edges are red, then A and B are indistinguishable – they are the same square even though they look different at first.

   Suppose you have n possible colors to use for painting the edges. Use Burnside's formula applied to D_4 to determine how many non-indistinguishable squares can be made. (For example, squares A and B would count together as one square.) Your answer will be a degree 4 polynomial in n.