MTH 236, Spring 2024 - Homework 8

Due on Friday, March 22 at 11:59pm on gradescope

1. Let G be a group and let N be a normal subgroup of G. Are the following statements true or false? Justify your answers, with examples where necessary.

   (a) If G is finite, then G/N is finite.

   (b) If G/N is finite, then G is finite.

   (c) If G is abelian, then G/N is abelian.

   (d) If G/N is abelian, then G is abelian.

2. Show that for $n \geq 2$, S_n is not a normal subgroup of S_{n+1}. (Hint: show that there is an element $\sigma \in S_n$ and some $\tau \in S_{n+1}$ such that $i_\tau(\sigma) = \tau\sigma\tau^{-1}$ is not in S_n. Try taking σ to be a cycle of length n, and τ to be a carefully chosen transposition.)

3. Let G be a group and let $K, H \leq G$. We say that H is conjugate to K if there exists $g \in G$ such that $gHg^{-1} = K$.

   (a) Prove that conjugacy is an equivalence relation on the set of all subgroups of G.

   (b) If H is normal in G, what is the equivalence class of H under this relation?

   (c) What are the equivalence classes of this relation for the set of subgroups of S_3? (There are precisely 6 subgroups of S_3.)

4. Let $\phi : G \to G'$ be a homomorphism. Show that $\phi[G]$ is abelian if and only if $xyx^{-1}y^{-1} \in \mathrm{Ker}(\phi)$ for all $x, y \in G$.

5. Let G be a finite group and let N be a normal subgroup of G of index r (so $|G/N| = r$.) Prove that $x^r \in N$ for every $x \in G$.

6. Let $G = \mathrm{GL}_2(\mathbb{R})$ and let $H = \{A \in G \mid \det(A) = 1\}$. (This group is called $\mathrm{SL}_2(\mathbb{R})$, the 2-dimensional *special linear group*.)

   (a) Show that H is normal in G.

   (b) Show that G/H is isomorphic to $\mathbb{R}^*$.

7. Let F be the group of all functions $f : \mathbb{R} \to \mathbb{R}$ under addition.

(a) Let $H \leq F$ be the subgroup of all functions f such that $f(0) = 0$. What group is F/H isomorphic to? (Hint: what is H the kernel of?)

(b) Let $C \leq F$ be the subgroup of constant functions. Show that F/C is isomorphic to the subgroup H from part (a).

(c) Let $K \leq F$ be the subgroup of all functions f that are continuous everywhere. Is there an element of F/K of order 2? Why or why not?

8. Let G be the group of functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$ where $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$, under the operation of function composition. (You should convince yourself that G is a group, but you don't have to show this.)

(a) Let $H = \{f \in G \mid f(x) = ax\}$. Show that H is not a normal subgroup of G.

(b) Let $K = \{f \in G \mid f(x) = x + b\}$. Show that K is a normal subgroup of G.

(c) Show that G/K is isomorphic to $\mathbb{R}^*$.

9. (a) Let $G = \mathbb{Z}_{18} \times \mathbb{Z}_{24}$. Find the orders of the groups $G/\langle(1,1)\rangle$ and $G/(\langle 12\rangle \times \langle 10\rangle)$.

(b) Find the orders of $(1,7) + \langle(1,1)\rangle$ in $\mathbb{Z}_6 \times \mathbb{Z}_9/\langle(1,1)\rangle$ and $(2,1) + \langle(2,3)\rangle$ in $\mathbb{Z}_6 \times \mathbb{Z}_9/\langle(2,3)\rangle$.

(c) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. Let $H_1 = \langle(2,1)\rangle$ and $H_2 = \langle(2,0)\rangle$. Note that H_1 and H_2 are isomorphic (they are groups of order 2). The groups G/H_1 and G/H_2 are order 4, and so are isomorphic to either $\mathbb{Z}_4$ or the Klein 4-group V. Compute which one each is isomorphic to.