MTH 236, Spring 2024 - Homework 7

Due on Friday, March 8 at 11:59pm on gradescope, but may be handed in through Friday, March 15th without penalty

1. True or false? Provide brief justifications for your answers.

   (a) For any groups A and B, $A \times B$ is isomorphic to $B \times A$.

   (b) $\mathbb{Z}_m \times \mathbb{Z}_n$ has an element of order mn only when m and n are relatively prime.

   (c) $\mathbb{Z}_m \times \mathbb{Z}_n$ has order mn only when m and n are relatively prime.

   (d) The Klein 4 group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

   (e) Both $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_5$ are cyclic.

   (f) Every abelian group of square free order is cyclic. (We say a number is square free if it is not divisible by p^2 for any prime p.)

   (g) Every abelian group of order a power of a prime is cyclic.

   (h) If 7 divides the order of a finite abelian group G, then G contains a subgroup of order 7.

   (i) If 7 divides the order of a finite abelian group G, then G contains a cyclic subgroup of order 7.

   (j) If 35 divides the order of a finite abelian group G, then G contains a cyclic subgroup of order 35.

   (k) If 36 divides the order of a finite abelian group G, then G contains a cyclic subgroup of order 36.

   (l) $\mathbb{Z}_{10} \times \mathbb{Z}_{12}$ is isomorphic to S_5.

   (m) $S_3 \times S_4$ is not isomorphic to a subgroup of S_6.

   (n) Let G and G' be arbitrary groups. There exists a homomorphism $\theta : G \to G'$.

   (o) The kernel of every homomorphism is non-empty (i.e. not the empty set).

   (p) There is a non-trivial homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_7$.

   (q) There is a non-trivial homomorphism $\psi : \mathbb{Z}_7 \to \mathbb{Z}$.

   (r) There is a non-trivial homomorphism $\chi : \mathbb{Z}_7 \to S_5$.

2. Let G be a group with subgroups H and K. Suppose that $H \cap K = \{e\}$, that every $g \in G$ can be written as $g = hk$ for some $h \in H$ and $k \in K$, and that $hk = kh$ for every $h \in H$ and $k \in K$. Show that G is isomorphic to $H \times K$.

3. (a) For each of the following you are given a finite abelian group G and an element $a \in G$. Find the order of a in G.

   i. $(2, 2, 2)$ in $\mathbb{Z}_3 \times \mathbb{Z}_{10} \times \mathbb{Z}_5$.

   ii. $(2, 5, 9)$ in $\mathbb{Z}_3 \times \mathbb{Z}_{10} \times \mathbb{Z}_{15}$.

   (b) What is the largest order of an element of $\mathbb{Z}_2 \times \mathbb{Z}_4$? How about $\mathbb{Z}_{12} \times \mathbb{Z}_9$? Justify both answers briefly.

   (c) Does the group $\mathbb{Z}_{12} \times \mathbb{Z} \times \mathbb{Z}_5$ have any nontrivial elements of finite order? Justify your answer.

   (d) How many subgroups of order 2 does $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_6$ have? Justify your answer.

   (e) Find all abelian groups of order 36, 40, 36^2, and 72 (up to isomorphism).

   (f) How many non-isomorphic abelian groups are there of order 36^3 and of order 1700?

   (g) Let p be a prime number. Find all abelian groups (up to isomorphism) of order p^4.

4. (a) Note that $\mathbb{Z}_3 \times \mathbb{Z}_{30} \times \mathbb{Z}_{20}$ and $\mathbb{Z}_6 \times Z_6 \times \mathbb{Z}_{50}$ have the same number of elements. Are they isomorphic? Why or why not?

   (b) Answer the same question for $\mathbb{Z}_{12} \times \mathbb{Z}_{70} \times \mathbb{Z}_{10}$ and $\mathbb{Z}_{30} \times \mathbb{Z}_{20} \times \mathbb{Z}_{14}$.

5. Let G be a group. Define the *center* of G to be

$$\mathbf{Z}(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\},$$

that is, the set of all elements of G that commute with every element of G. Observe that G is abelian if and only if $\mathbf{Z}(G) = G$.

(a) Show that $\mathbf{Z}(G)$ is a subgroup of G.

(b) Show that $\mathbf{Z}(G)$ is precisely the set of elements $x \in G$ such that conjugation $c_x : G \to G$ is the identity function.

(c) If $\phi : G \to H$ is an isomorphism, show that $\phi[\mathbf{Z}(G)] = \mathbf{Z}(H)$.

(d) S_n is nonabelian for $n \geq 3$. Now show that if $n \geq 3$, then $\mathbf{Z}(S_n)$ is the trivial subgroup of S_n. (Hint: if σ is a single cycle, it is easy to find some τ that does not commute with σ. For an arbitrary σ, write it as a product of cycles, and use the fact that disjoint cycles commute.)

(e) Let G be a group and $H \leq G$ be any subgroup. Is it always true that $\mathbf{Z}(H) = \mathbf{Z}(G) \cap H$? Either prove this or give a counterexample.

6. A subgroup $H \leq G$ is called *normal* if the set of left cosets of H is the same as the set of right cosets of H.

   (a) Show that if H is contained in $\mathbf{Z}(G)$ (the center of G), then H is normal. (So if G is abelian, every subgroup is normal.)

   (b) Give an example of a group G and a subgroup H that is not normal.

   (c) Show that for any group G, if $|G : H| = 2$, then H is normal.

7. For each of the following you are given two groups G and G' and map $\phi : G \mapsto G'$. Determine (with proof) whether ϕ is a homomorphism. If it is, describe the kernel of ϕ.

   (a) $\phi : \mathbb{R} \to \mathbb{Z}$, where $\phi(x)$ is the smallest integer greater than or equal to x. (This map is often called the *ceiling function*.)

   (b) $\phi : \mathbb{R}^+ \to \mathbb{R}$, where $\phi(x) = \ln(x)$. ($\mathbb{R}^+$ is a group under multiplication and $\mathbb{R}$ is a group under addition.)

   (c) Let $M_n(\mathbb{Q})$ be the additive group of $n \times n$ matrices with values in $\mathbb{Q}$, and let $\mathbb{Q} = \langle \mathbb{Q}, + \rangle$. Let $\phi : M_n(\mathbb{Q}) \to \mathbb{Q}$ be given by $\phi(A) = \det(A)$.

   (d) $\phi : M_n(\mathbb{Q}) \to \mathbb{Q}$ given by $\phi(A) = \mathrm{tr}(A)$. (Recall that $\mathrm{tr}(A)$ denotes the trace of A, i.e. the sum of its diagonal entries.)

8. For each of the following you are given two groups G and G' and a homomorphism $\theta : G \mapsto G'$. Determine the indicated quantities.

   (a) Find the kernel of θ, and also find $\theta((-5, 3))$, if you know that $\theta : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ satisfies $\theta((1, 0)) = 4$ and $\theta((0, 1)) = 6$.

   (b) Find the kernel of θ, and also find $\theta((-1, 5))$ and $\theta((3, -1))$, if you know that $\theta : \mathbb{Z} \times \mathbb{Z} \to S_9$ satisfies $\theta((1, 0)) = (1, 2)(3, 4)$ and $\theta((0, 1)) = (5, 6, 7, 8, 9)$.

9. Let G and G' be finite groups and $\theta : G \mapsto G'$ a homomorphism.

   (a) Suppose that $|G'|$ is prime. Show that θ must either be trivial or surjective.

   (b) Suppose that $|G|$ is prime. Show that θ must either be trivial or injective.

   (c) Suppose that $|G|$ and $|G'|$ are the same prime number. Show that θ is either trivial or an isomorphism.

   (d) Suppose that $|G|$ and $|G'|$ are relatively prime. Show that θ must be trivial.