MTH 236, Spring 2024 - Homework 6

Due on Friday, March 1 at 11:59pm on gradescope

Question 1. Let $S_{\mathbb{Z}}$ be the group of all permutations of $\mathbb{Z}$.

(a) Prove that every finite group G is isomorphic to a subgroup of $S_{\mathbb{Z}}$. (Hint: this proof should be very short if you appeal to the correct theorem...)

(b) Find a subgroup of $S_{\mathbb{Z}}$ that is isomorphic to $\mathbb{Z}$. (This shows how ridiculously large $S_{\mathbb{Z}}$ is; it contains every finite group as a subgroup, but has even more subgroups than that.)

(c) Let $H = \{\sigma \in S_{\mathbb{Z}} \mid \sigma(x) = x \text{ for all but finitely many } x \in \mathbb{Z}\}$, that is, H consists of the set of permutations that move only finitely many elements. Show that H is a subgroup of $S_{\mathbb{Z}}$.

(d) Find all of the orbits of $\sigma, \tau \in S_{\mathbb{Z}}$ where $\sigma(x) = 2 - x$ and $\tau(x) = x + 3$.

Question 2. Let $\sigma \in S_n$ be written as $\sigma = \mu_1\mu_2 \dots \mu_k$ where $\mu_1, \mu_2, \dots, \mu_k$ are disjoint cycles of lengths $\ell_1, \ell_2, \dots, \ell_k$. Show that the order of σ is the LCM (least common multiple) of $\ell_1, \ell_2, \dots, \ell_k$. (This is the smallest positive integer m such that $\ell_1, \dots, \ell_k$ are all divisors of m.)

Question 3. In the proof of Cayley's Theorem we used the left multiplication permutation $\lambda_x : G \to G$, defined by $\lambda_x(a) = xa$. Another way for elements to permute the group they belong to is by *conjugation*: define $c_x : G \to G$ by $c_x(g) = xgx^{-1}$.

(a) Show that $c_x \in S_G$ for every $x \in G$.

(b) Unfortunately, this permutation cannot be used to prove Cayley's Theorem: define the map $\phi : G \to S_G$ by $\phi(x) = c_x$. Show that ϕ is a homomorphism, but give an example to show that ϕ is not necessarily injective.

Question 4. Let $\phi : G \to H$ be a homomorphism. The *kernel* of ϕ is

$$\ker \phi = \{x \in G \mid \phi(x) = e_H\}$$

where e_H is the identity of the group H.

(a) Show that $\ker \phi$ is a subgroup of G.

(b) Show that if $g \in \ker \phi$, then $c_x(g) = xgx^{-1} \in \ker \phi$ for every $x \in G$.

(c) Show that ϕ is injective if and only if $\ker \phi$ is the trivial subgroup of G.

Question 5. Let G be a group. An *automorphism* of G is an isomorphism $\phi : G \to G$. The *automorphism group* $\mathrm{Aut}(G)$ is the group of all automorphisms of G under the operation of function composition.

(a) Show that $\mathrm{Aut}(G)$ is a subgroup of S_G (in particular, this shows that $\mathrm{Aut}(G)$ *is* a group).

(b) Show that for every $x \in G$, the function c_x is an automorphism of G, where c_x is defined as in Question 5. (You already did part of this in 5(a).)

(c) Show that if $G = \langle a \rangle$ is cyclic and $\phi \in \mathrm{Aut}(G)$, then $\phi(a)$ is a generator of G.

(d) Use part (c) to compute $\mathrm{Aut}(G)$ for $G = \mathbb{Z}_4$ and $G = \mathbb{Z}_5$.