MTH 236, Spring 2024 - Homework 5

Due on Sunday, February 25 at 11:59pm on gradescope

For any group G, recall that the *trivial subgroup* of G is the subgroup containing only the identity, and a *proper subgroup* of G is any subgroup other than G itself.

Question 1. (a) Suppose that a is an element of some group and the order of a is 105. Find the orders of each of the following elements: a^{25}, a^{44}, a^{70}.

(b) Suppose that $G = \langle a \rangle$ is a cyclic group of order 6000. Find all of the elements of G that have order 6.

(c) Suppose that a cyclic group $G = \langle a \rangle$ has exactly one nontrivial proper subgroup and that subgroup has order 11. What is the order of G?

(d) Let p and q be *distinct* primes. How many subgroups does a cyclic group of order pq have? How many generators does it have?

(e) Let p be prime and let $n \geq 1$. How many subgroups does a cyclic group of order p^n have? How many generators does it have?

Question 2. Prove that a group G equals the union of all of its proper subgroups if and only if G is *not* cyclic.

Question 3. It is obvious that if a group G is abelian, then all of its proper subgroups are abelian (you don't have to prove this). Show that the converse to this fails: there exists a group G with all proper subgroups abelian such that G itself is not abelian.

Question 4. Show that the group S_3 is generated by two of its elements. That is, find two elements $\sigma, \tau \in S_3$ such that $S_3 = \langle \sigma, \tau \rangle$, then explicitly write every element of S_3 as some product of powers of σ and τ.

Question 5. Let A be a set and let $G = S_A$ be the group of all permutations of A. Let $x \in A$ and define

$$G_x = \{\sigma \in G \mid \sigma(x) = x\}.$$

Prove that G_x is a subgroup of G.

Question 6. Find a subgroup of S_4 that is isomorphic to the Klein 4-group V.

Question 7. The following six matrices

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

form a group under matrix multiplication. Give an isomorphism between the matrix group and a familiar group. You do not have to prove that your map is an isomorphism.

Hint: Try seeing what effect each matrix has on $\begin{bmatrix}1\\2\\3\end{bmatrix}$.

Question 8. The following permutations are elements of S_5.

$$\sigma = \begin{pmatrix}1&2&3&4&5\\2&1&3&4&5\end{pmatrix}, \quad \tau = \begin{pmatrix}1&2&3&4&5\\1&2&4&5&3\end{pmatrix}.$$

Show that $H = \langle \sigma, \tau \rangle$ is a cyclic group of order 6. Find a generator for H.

Question 9. (a) Show that S_n is nonabelian for $n \geq 3$.

(b) Let H be any subgroup of S_n. Show that either all of the elements of H are even or exactly half of the elements of H are even. (Hint: mimic the proof that $|A_n| = |S_n|/2$.)

Question 10. The following permutation is an element of S_{10}.

$$\sigma = \begin{pmatrix}1&2&3&4&5&6&7&8&9&10\\2&1&4&5&3&7&8&9&10&6\end{pmatrix}$$

(a) Find all the orbits of σ and write σ as a product of disjoint cycles.

(b) Is σ even or odd? (Of course, justify your answer.)

(c) What is the order of σ?