MTH 236, Spring 2024 - Homework 3

Due on Friday, February 9 at 11:59pm on gradescope

Question 1. True or false? Justify your answers.

1. Any two groups of order 3 are isomorphic.
2. Any two groups of order 4 are isomorphic.
3. Every group of at most four elements is abelian.

Question 2. Are the following groups isomorphic? Either way, give a proof. If the groups are isomorphic, write down an explicit isomorphism.

1. $\langle \mathbb{Q}, + \rangle$ and $\langle \mathbb{R}, + \rangle$
2. $\langle \mathbb{R}, + \rangle$ and $\langle \mathbb{R}^*, \cdot \rangle$
3. $\langle \mathbb{C}^*, \cdot \rangle$ and $\langle \mathbb{R}^*, \cdot \rangle$ (Fact: $|\mathbb{C}^*| = |\mathbb{R}^*|$)
4. $\langle \mathbb{R}, + \rangle$ and $\langle \mathbb{R}^+, \cdot \rangle$ (Hint: do you know a function from calculus that changes multiplication into addition, or vice versa?)
5. G and H, where G is the set of (infinite) sequences of integers under the operation $+$, that is, $a \in G$ is of the form $a = (a_1, a_2, a_3, \dots)$ where each $a_i \in \mathbb{Z}$, and

$$(a_1, a_2, a_3, \dots) + (b_1, b_2, b_3, \dots) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots),$$

   and H is the set of functions $f : \mathbb{Q} \to \mathbb{Z}$ under the operation $+$ (addition of functions).

Question 3. Let G be a group and H_1 and H_2 be subgroups of G.

1. Must the intersection $H_1 \cap H_2$ be a subgroup of G? If true give a proof. If false, give a counterexample.
2. Answer the same question for the union $H_1 \cup H_2$.

Question 4. Let G be an abelian group.

1. $H = \{a^2 \mid a \in G\}$ be the set of all squares in G. Prove that H is a subgroup of G.
2. Let $H = \{a \in G \mid a^2 = e\}$. Prove that H is a subgroup of G.

Question 5. Let G be a group and H a subset of G. Define a relation $\sim$ on G by $a \sim b$ if and only if $a^{-1}b \in H$. Suppose that H is a subgroup of G and prove that $\sim$ is an equivalence relation on G. What is the equivalence class of the identity element? (Recall that the equivalence class of x is the set $\bar{x} = \{a \mid a \sim x\}$.)

Question 6. Let G be a finite group and $a \in G$. Prove that $a^n = e$ for some integer $n \geq 1$.

Question 7. Let G be a group with a nonempty finite subset H that is closed under multiplication (that is, under the group operation).

(a) Prove that H must be a subgroup of G.

(b) Give an example to show that the result is no longer true if we do not assume that the subset H is finite.

Question 8. Let G be a group and suppose that $g \in G$ is the unique element of G other than the identity e that equals its own inverse. Prove that g commutes with all elements of G, that is, $ga = ag$ for all $x \in G$. [Hint: Show that $aga^{-1} \neq e$ for any $a \in G$, then find the inverse of aga^{-1}.]

Question 9. Let $m, n \in \mathbb{Z}$, and define $H_{m,n}$ to be the set of all possible "linear combinations" of m and n, so that $H_{m,n} = \{a\,m + bn \,|\, a, b \in \mathbb{Z}\}$.

(a) Prove that $H_{m,n}$ is a subgroup of $\langle \mathbb{Z}, + \rangle$.

(b) Prove that if K is any subgroup of $\langle \mathbb{Z}, + \rangle$ such that $m \in K$ and $n \in K$, then $H_{m,n} \leq K$. (This shows that $H_{m,n}$ is the smallest subgroup containing both m and n.)