MTH 236, Spring 2024 - Homework 2

Due on Friday, February 2 at 11:59pm on gradescope

Question 1. Let S be a set with n elements. Explain your answers to the following.

(a) How many binary operations can be put on S?

(b) How many commutative binary operations can be put on S?

Question 2. The table below can be completed to define an associative binary operation $*$ on $\{a, b, c, d\}$. Compute the missing entries, showing all your reasoning. (The way to read the table is that $x * y$ is found in the row corresponding to x on the left and y on top. For example, $b * d = c$ and $d * b = b$.)

$*$	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c				
d	d	b	c	a

Question 3. For each of the following you are given a set A and a binary operation $*$. Determine whether the operation is commutative and whether it is associative (for each property, either show that it has that property or find a counterexample). You may use that the usual addition and multiplication on $\mathbb{R}$ (and therefore all subsets of $\mathbb{R}$) are commutative and associative.

(a) $A = \mathbb{Q}$, $*$ defined by $a * b = 5ab$.

(b) $A = \mathbb{Q}$, $*$ defined by $a * b = 5ab - 1$.

(c) $A = \mathbb{Z}$, $*$ defined by $a * b = a - b + 1$

(d) $A = \mathbb{N}$, $*$ defined by $a * b = 3^{a+b}$.

Question 4. Is it true that every commutative binary operation on a set $A = \{a, b\}$ with exactly two elements must also be associative? Give a proof or a counterexample.

Question 5 True or false? Justify your answers.

1. In a group $\langle G, * \rangle$ for any $a \in G$, there exists $x \in G$ that satisfies $x * x = a$.
2. In the group table for a group $\langle G, * \rangle$, each element of G appears exactly once in each row and exactly once in each column.

Question 6 For each of the following, determine whether the set G is a group under the operation $*$. Justify your answers. $M_n(\mathbb{R})$ is the set of $n \times n$ matrices with entries in $\mathbb{R}$, $\det(A)$ is the determinant of the matrix A, and I_n is the $n \times n$ identity matrix. You may assume the fact from linear algebra that multiplication of matrices is associative (as well as any other facts you know about matrices and determinants).

1. $G = 3\mathbb{Z} = \{3n | n \in \mathbb{Z}\}$, $a * b = a + b$.
2. $G = 3\mathbb{Z}$, $a * b = a + 2b$.
3. $G = \mathbb{R}^*$ (the nonzero real numbers), $a * b = |ab|$.
4. $G = \{c + d\sqrt{2} : c \in \mathbb{Q}, d \in \mathbb{Q},$ and c, d are not both zero$\}$, $a * b = ab$.
5. $G = \mathbb{R}^+$, $a * b = \sqrt{ab}$.
6. $G = M_2(\mathbb{R})$, $A * B = AB$ (multiplication of matrices).
7. $G = \{A \in M_2(\mathbb{R}) \mid \det(A) = 1\}$, $A * B = AB$.
8. $G = \{A \in M_2(\mathbb{R}) \mid A^2 = I_2\}$, $A * B = AB$.

Question 7

1. Let $\langle G, * \rangle$ be a group in which every element of G is its own inverse. Prove that G is abelian (that is, $*$ is commutative).
2. Let $\langle G, * \rangle$ be a finite group with an even number of elements. Prove that there exists an element $a \neq e$ in G that equals its own inverse.