MTH 236, Spring 2024 - Homework 11

Due on April 12th at 11:59pm on gradescope

1. Let $S \subseteq \mathbb{R}$ be defined by $S = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$.

   (a) Show that S is a ring under the usual addition and multiplication.

   (b) Show that S is not a field.

   (c) Show that if $a + b\sqrt{3} = c + d\sqrt{3}$ for $a, b, c, d \in \mathbb{Z}$, then $a = c$ and $b = d$. (Use the fact that $\sqrt{3}$ is irrational.)

   (d) Show that the units of S are precisely the elements $a + b\sqrt{3}$ with $a^2 - 3b^2 = \pm 1$. (Suppose $(a + b\sqrt{3})(c + d\sqrt{3}) = 1$. Using part (c), show that $ad + bc = 0$. Then compute $(a - b\sqrt{3})(c - d\sqrt{3})$. Finally, show that $(a^2 - 3b^2)(c^2 - 3d^2) = 1$, and use what you know about the units of $\mathbb{Z}$.)

   (e) Note that $R' = \left\{ \begin{bmatrix} a & 3b \\ b & a \end{bmatrix} \middle| \ a, b \in \mathbb{Z} \right\}$ in a ring under the usual operations of addition and multiplication. You are not asked to confirm this. Prove that the map $\phi : R \mapsto R'$ defined by $\phi(a + b\sqrt{3}) = \begin{bmatrix} a & 3b \\ b & a \end{bmatrix}$ is a ring isomorphism.

2. (a) Find all the units for each of the following rings. Justify your answers briefly.

      i. $\mathbb{Z}_{15}$.

      ii. $\mathbb{Z}_{11}$.

      iii. $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}_3$.

   (b) How many units are in $M_2(\mathbb{Z}_3)$, the ring of all 2×2 matrices with entries in $\mathbb{Z}_3$? Briefly justify your answer.

   (c) Let R be a ring with multiplicative identity and U be the set of all units in R. Prove that U is a group under multiplication.

   (d) Prove that for any ring R, no element $a \in R$ is both a unit and a zero divisor.

   (e) Show that if $ab = 1$ in a ring R with multiplicative identity 1 and no zero divisors, then $ba = 1$ also (even if the ring is not a commutative ring).

3. Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

4. For each of the following, find all solutions of the given equation in the given ring or show there are no solutions.

   (i) $4x = 2$ in $\mathbb{Z}_{13}$.

   (ii) $4x = 2$ in $\mathbb{Z}_8$.

   (iii) $x^2 + 4x - 2 = 0$ in $\mathbb{Z}_6$.

   (iv) $x^2 - 1 = 0$ in $\mathbb{Z}_8$.

   (v) $x^2 + 4x + 3 = 0$ in $\mathbb{Z}_{15}$.

5. True or false? Provide brief justifications for your answers.

   (a) $a^{p-1} \equiv 0 \pmod{p}$ for all integers a and primes p.

   (b) $\varphi(n)$ can never equal n for a positive integer $n \geq 2$ (where $\varphi(n)$ represents the Euler φ function).

   (c) Let $n \geq 2$ be an integer. The units in $\mathbb{Z}_n$ are all numbers in the set $\{1, 2, 3, \ldots, n-1\}$ which are relatively prime to n.

   (d) In a commutative ring with unity, the product of two units is always a unit.

   (e) In a commutative ring with unity, the product of two non-units is always a non-unit.

   (f) In a commutative ring with unity, the product of a unit and a non-unit is never a unit.

6. Determine the characteristic of each of the following rings.

   (i) $\mathbb{Z} \times \mathbb{Q}$.

   (ii) $\mathbb{Z}_4 \times \mathbb{Z}_5$.

   (iii) $\mathbb{Z}_4 \times \mathbb{Z}_6$.

7. Let p be a prime, in which case $\mathbb{Z}_p$ is a field. It can be shown that the group $\mathbb{Z}_p^*$ of nonzero elements of $\mathbb{Z}_p$ under multiplication is actually cyclic. You do not have to prove this in general, but confirm this in the special case $p = 13$ by finding a generator for the group $\mathbb{Z}_{13}^*$ and justifying your answer.

8. Find all positive integers n such that $\mathbb{Z}_n$ contains a subring isomorphic to $\mathbb{Z}_2$. Justify your answer.

9. Let R be the ring of all *smooth* functions from $\mathbb{R}$ to $\mathbb{R}$ (i.e. all functions from $\mathbb{R}$ to $\mathbb{R}$ that have derivatives of all orders) under the usual multiplication and addition. Is the map $\Phi : R \to R$ defined by $\Phi(f)(x) = f'(x)$ a homomorphism of the underlying additive groups? Is it also a ring homomorphism? Why or why not?