Math 236: Abstract Algebra

Midterm II

April 2, 2024

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Your University email ______________________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ______________________________

Instructions:

- **The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden. You must be physically separated from your cell phone.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 6 pages.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
TOTAL	50	

1. (10 points)

(a) Consider the group $G = \mathbb{Z}_7 \times \mathbb{Z}_{12}$. Let K be the subgroup generated by $(1,2)$ in G and find the order of the element $(1,1) + K$ in G/K.

Sol. $|G| = 12 * 7 = 84$, $|K| = lcm(7,6) = 42$, so that $|G/K| = 84/42 = 2$. Hence, the order is 1 or 2, but $(1,1) \notin K$, as the second co-ordinate of element of K is even mod 12, being $2n$ modulo 12. So the order is 2.

There are various other ways to see the whole answer or just the last part: (i) Order of $(1,1)$ is lcm $(7,12) = 84$, so that $(1,1) \notin K$, (ii) $(2,2) = 37(1,2) \in K$, but $(1,1) \notin K$ as above. So the order is 2. (iii) $(1,1) - (1,2) = (0,-1)$ gives another coset representative, which is not in K, as above, but $2(0,-1) = -7(1,2)$, so that the order is 2.

(b) Find all abelian groups of order 8 and order 36 up to isomorphism. Indicate which ones on your lists are cyclic.

Sol. (i) Z_8 cyclic, $Z_4 \times Z_2$ and $Z_2 \times Z_2 \times Z_2$.

(ii) Z_{36} cyclic, $Z_2 \times Z_2 \times Z_9$, $Z_4 \times Z_3 \times Z_3$, $Z_2 \times Z_2 \times Z_3 \times Z_3$.

(Note $Z_{36} = Z_4 \times Z_9$, the second is $Z_2 \times Z_{18}$ etc.)

If you use the invariant factors, the answer (i) would look the same, but the answer (ii) would be Z_{36} cyclic, $Z_2 \times Z_{18}$, $Z_3 \times Z_{12}$, $Z_6 \times Z_6$.

2. (10 points)

(a) Show that if H is a subgroup of a group G and N is a normal subgroup of G then $H \cap N$ is a normal subgroup of H. (You may assume $H \cap N$ is a subgroup of H, so you just need to show it is normal.)

Sol. If $x \in H \cap N$, and $h \in H$, then hxh^{-1} is in N, as N is normal, and it is also in H by the closure properties for the inverse and the multiplication for the group H. So, it is in the intersection, which is thus normal in H.

Note: The intersection need not be normal in G.

(b) Show that every normal subgroup of G having only two elements is contained in $Z(G)$.

Sol. If $N = \{e, n\}$ is a normal subgroup of G with 2 elements, e being the identity, then $e \in Z(G)$ as usual, and normality implies that for given any $g \in G$, gng^{-1} is in N, and as it cannot be e (by cancellation law), it has to be n forcing $gn = ng$ for all $g \in G$. Thus $n \in Z(G)$. Thus, $N \subset Z(G)$.

(c) Prove that if N is a proper nontrivial normal subgroup of S_{10} then $N = A_{10}$. (Hint: You may use that A_n is simple for $n \geq 5$.)

Sol. Assume N is not A_{10}. By part (a), $N \cap A_{10}$ is normal in A_{10} and is thus trivial by simplicity of A_{10}. Thus N contains an odd permutation. Product of any two odd permutations in N is even, and so has to be e, by triviality above. So the intersection has to contain exactly 2 elements, and thus by part (b), it is in center of S_{10}, which is trivial, a contradiction. (We need not use the fact that the center is trivial: N being normal, it contains all elements of the same cycle type as any given element in it, so cannot just contain two elements, as there is one element of it of order 2).

3. (10 points) Circle **True** or **False**. No justification needed.

(a) **True False** Suppose G and G' are finite groups with $|G| = p$, prime and $|G'| = q$, prime. If $\phi : G \to G'$ is a group homomorphism then p must equal q and ϕ must be an isomorphism.

Sol. F. ϕ can be trivial.

(b) **True False** If G is a finite abelian group and d is a divisor of $|G|$ then G has a subgroup of order d.

Sol. T. Theorem we proved, as a corollary of structure theorem for finite abelian groups.

(c) **True False** A homomorphism is an injection if and only if its kernel contains only the identity. **Sol.** T. Theorem we proved.

(d) **True False** It is possible to have a one-to-one homomorphism from some group of order 9 into some group of order 12.

Sol. F. Its image would be an order 9 subgroup of an order 12 group, contradicting Lagrange's theorem.

(e) **True False** Given a group G of order 90, it is always possible to have a one-to-one homomorphism from any group of order 9 into G.

Sol. F. For example, $G = Z_3 \times Z_3 \times Z_{10}$ does not have Z_9 as subgroup.

4. (10 points)

(a) Define a conjugacy class of an element g_0 in a group G. If G is finite, explain why the number of elements in it divides the order of G.

Sol. The conjugacy class c in G of g_0 is defined to be $c = \{gg_0g^{-1} : g \in G\}$. It being the orbit of g_0 under the conjugacy action of G on G, by the orbit-stabilizer theorem, its order divides the order of G.

Note : $\{g_0gg_0^{-1} : g \in G\}$ is the whole group G and not the conjugacy class.

(b) Suppose that G is an abelian group of order $9n$, where n is not divisible by 3, and G has exactly a elements of order 3. Find all possibilities values of a. Justify.

Sol. By the structure theorem of finite abelian groups, G is isomorphic to $Z_9 \times G_n$ or to $Z_3 \times Z_3 \times G_n$ for some abelian group G_n of order n, and thus the lcm formula for orders of elements in direct products, we see that a is thus the number of elements of order 3 in either Z_9 (in which case $a = 2$, the elements being 3 or 6) or in $Z_3 \times Z_3$ (in which case $a = 8$, elements being any non-identity elements in it). Thus $a = 2$ or 8.

5. (10 points)

(a) Find the orders of $(2,0,2)$ and $(2,1,2)$ in $\mathbb{Z}_4 \times \mathbb{Z} \times \mathbb{Z}_7$.

Sol. $\text{ord}(2,0,2) = lcm(4/gcd(2,4), 1, 7/gcd(2,7)) = lcm(2,7) = 14$. Since (the second component) 1 is an element of infinite order in Z, tghe second element has infinite order.

(b) Find the kernel of θ, and also find $\theta((2,-2))$, if you know that the homomorphism $\theta : \mathbb{Z} \times \mathbb{Z} \to S_7$ satisfies $\theta((1,0)) = (1,2,3)$ and $\theta((0,1)) = (4,5)(6,7)$.

Sol. Note that since the given permutations are disjoint, they commute and thus by the homomorphism property, $\theta((a,b)) = (123)^a(45)^b(67)^b = e$ if and only if 3 divides a and 2 divides b. Thus, kernel θ is $3Z \times 2Z = \{(3n, 2m) : n, m \in Z\}$.

$\theta((2,-2)) = (123)^2(45)^{-2}(67)^{-2} = (123)^2 = (132)$.

Note: The kernel is not $2Z \times 3Z$ and it is not $< (3,2) >= \{(3n, 2n) : n \in Z\}$.