Math 236: Abstract Algebra

Midterm I

February 27th, 2024

NAME (please print legibly): SOLUTIONS
Your University ID Number: ______________________
Your University email ______________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

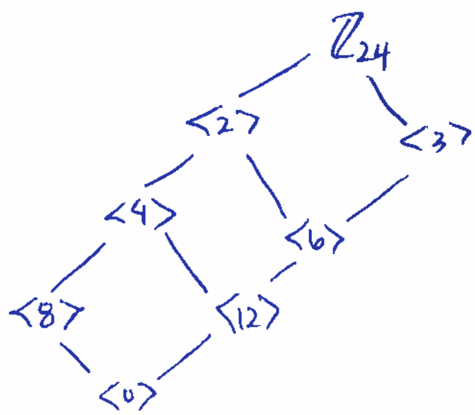
Signature: ______________________

Instructions:

- **The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden. You must be physically separated from your cell phone.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 8 pages.**

1. (10 points) (No jutifications needed)

(a) Draw a subgroup diagram for $\mathbb{Z}_{24}$.



(b) Find all the distinct generators of $\mathbb{Z}_{24}$.

$1, 5, 7, 11, 13, 17, 19, 23$

(c) Find all the distinct elements of $\langle 20\rangle \leq \mathbb{Z}_{24}$.

$\langle 20\rangle = \langle 4\rangle = \{0, 4, 8, 12, 16, 20\}$

$\uparrow$

$\gcd(20, 24) = 4$

(d) Find all possible distinct generators of $\langle 20 \rangle \leq \mathbb{Z}_{24}$. $|\langle 20 \rangle| = 6$ & $\varphi(6) = 2$

$4, 20$

(e) Find a generator for the subgroup of $\mathbb{Z}_{24}$ generated by 6 and 20.

$\gcd(6, 20) = 2$

$\langle 6, 20 \rangle = \langle 2 \rangle \leq \mathbb{Z}_{24}$

2. (10 points) Suppose

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

(a) Compute $\sigma \circ \tau$ and $\tau \circ \sigma$.

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

(b) Compute σ^2 and τ^2.

$\sigma^2 = id$

$\tau^2 = id$

(c) Find $\langle \sigma, \tau \rangle \leq S_4$.

$\langle \sigma, \tau \rangle = \{ id, \sigma, \tau, \sigma\tau \}$

(d) In part (c), you found a 4-element group. Is it isomorphic to the cyclic group of 4 elements or the Klein 4-group? Justify!

V_4 since all ~~elemen~~ elements in $\langle \sigma, \tau \rangle$ square to the identity

(e) Write down disjoint cycle decompositions for each of σ and τ. (No explanation needed.)

$\sigma = (1,3)(2,4)$

$\tau = (2,4)$

3. **(10 points)**

(a) Suppose p and q are distinct primes. How many generators does $\mathbb{Z}_{pq}$ have? How many subgroups does $\mathbb{Z}_{pq}$ have, and what are they?

$\varphi(pq) = (p-1)(q-1)$ generators

Subgroups: 4

$\langle 0 \rangle, \langle p \rangle, \langle q \rangle, \mathbb{Z}_{pq}$

(a unique subgp for every divisor of pq)

(b) Let G be a finite group with an even number of elements. Then show that G must contain an element of order 2.

Pair every element x with its inverse x^{-1} if $x \neq x^{-1}$. There will be an even number of paired elements of G. The unpaired elements in the group are therefore even in number (since G has an even number of elements). But $e = e^{-1}$ so it is unpaired, so the must be at least one non-identity element x s.t. $x = x^{-1}$ or $x^2 = id$ and x is $\therefore$ order 2.

4. (10 points)

(a) Given a set A of order a, state (i) how many functions from A to A are there?, (ii) How many relations on A (i.e., relations between A and A) are there? Justify briefly.

A function whose domain is A is a set of pairs $(b_1, b_2) \in A \times A$ such that $\forall a \in A$, a occurs as the first element of exactly one pair. The second entry (i.e., b_2) can be anything so there are a^a choices. (That is, a choices for b_2 for each of a pairs.

A relation is a subset of $A \times A$. There are $2^{a^2} = 2^{|A \times A|}$ subsets of $A \times A$ so there are 2^{a^2} relations

(b) Define a relation on $\mathbb{Z}$ by $x \sim y$ if and only if $x + 2y$ is divisible by 3. Prove or disprove that $\sim$ is an equivalence relation on $\mathbb{Z}$.

reflexive: $3 | (x + 2x)$ so $x \sim x$

symmetric: If $x \sim y$ then $3 | (x + 2y)$ so $3 | 2(x+2y)$ so $3 | (2x + 4y)$

so $3 | (2x + 4y - 3y)$ so $3 | (2x + y)$. Thus $y \sim x$.

transitive: If $x \sim y$ and $y \sim z$ then $3 | (x + 2y)$ and $3 | (y + 2z)$

So $3 | (x + 2y + y + 2z)$ so $3 | (x + 3y + 2z)$ so $3 | (x + 2z) \Rightarrow x \sim z$.

Thus $\sim$ is an equivalence relation.

5. (10 points) Circle **True** or **False**. (No justification needed)

(a) **True** **False** If m and n are relatively prime then $\langle m, n\rangle \leq \mathbb{Z}$ is a proper nontrivial subgroup of $\mathbb{Z}$.

$\langle m, n\rangle = H_{m,n} = \langle \gcd(m,n)\rangle = \langle 1\rangle = \mathbb{Z}$

if m and n are rel. prime

(b) **True** **False** For some binary operations $*$ on $\mathbb{Z}$ and $*'$ on $\mathbb{Q}$, the binary structures $(\mathbb{Z}, *)$ and $(\mathbb{Q}, *')$ are isomorphic.

true: let $\phi: \mathbb{Z} \to \mathbb{Q}$ be a bijection (same cardinality)

Define $a *' b = \phi(\phi^{-1}(a) + \phi^{-1}(b))$

(c) **True** **False** If all the proper subgroups of a group are cyclic then that group must be cyclic.

ex/ S_3 all proper subgps: $\left.\begin{matrix}\langle \sigma \rangle \\ \langle \tau \rangle \\ \{e\}\end{matrix}\right\}$ all cyclic

(d) **True** **False** Every group is isomorphic to a subgroup of S_n for some n.

not infinite groups

(e) **True** **False** A group cannot be isomorphic to any of its proper subgroups.

ex/ $\langle \mathbb{Z}, +\rangle \simeq \langle 2\mathbb{Z}, +\rangle$