Q2, HW 4

(1) Determine subspaces U and W of $\mathbb{R}^3$ such that $\dim(U) = m$, $\dim(W) = n$, $m \geq n$, and both $\dim(U \cap W) < n$ and $\dim(U + W) < m + n$.

(2) Suppose that U and W are subspaces of a finite-dimensional vector space V and $V = U + W$. Show that $\dim(U) + \dim(W) \geq \dim(V)$ with equality if and only if $U \cap W = \{0\}$.

(3) If $\dim(U) + \dim(W) = \dim(V)$, does it follow that $V = U + W$? Prove this is true or provide a counter example.

Solution:

(1) Sketch of solution: There are many choices. For example, $W_1 = \text{span}\{(1, 0, 0), (0, 1, 0)\}$ and $W_2 = \text{span}\{(1, 0, 0), (0, 0, 1)\}$.

(2) Since $V = U + W$, every $x \in V$ can be written as an element in U plus an element in V. Let β_U be a basis for U and β_W a basis for W. Then $\beta_U \cup \beta_W$ is a spanning set for V. If $\dim(V) = n$, the size of a spanning set must be at least n. Thus $\mid \beta_U \mid + \mid \beta_W \mid = \dim(U) + \dim(V) \geq \dim(V)$.

Suppose $\dim(U) + \dim(V) = n$. Then $\beta_U \cup \beta_W$ constitutes a spanning set for V of size n. Then it is a basis, so indpendent. If $x \in U \cap W$, $x = \sum a_i v_i = \sum b_i v_j$, where the $v_i \in \beta_U$ and the $v_j \in \beta_W$. Then $x - x = \sum a_i v_i - \sum b_i v_j = \mathbf{0}$. This representation of zero is non-trivial unless $x = 0$. Hence $U \cap W = \{\mathbf{0}\}$.

Conversely, suppose $U \cap W = \{\mathbf{0}\}$. Then $V = U \oplus W$., so every $x \in V$ can be written uniquely as $x = u + w$ where $u \in U$ and $w \in W$. Since u and w in turn can written exactly one way as linear combinations of vectors in β_U and β_W respectively, we have that $\beta_U \cup \beta_W$ is a basis for V. Hence $|\beta_U \cup \beta_W| = n$. Since $\beta_U \cap \beta_W$ is empty, $|\beta_U| + |\beta_W| = |\beta_U \cup \beta_W| = n$.

(3) No. Take for example $U = \text{span}\{(1, 0, 0), (0, 1, 0)\}$ and $W = \text{span}\{(1, 0, 0)\}$ in $\mathbb{R}^3$..

Q3, HW4

Suppose V is a vector space over F, and $\dim(V) = n$. Find subspaces $W_1, W_2, \ldots, W_n$ of V such that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

Solution: The idea here is that one could let W_i be the span of the ith vector in any basis for V. Then use part (2).