Homework 12

Due Friday, April 19, 2024 at 11:59pm on gradescope

Academic honesty expectations: Same as on previous homeworks. We remind you that internet searches are not permitted.

1. Are the following matrices diagonalizable? If not, explain why. If yes, diagonalize A by finding Q such that $D = Q^{-1}AQ$ is diagonal (you must state what Q and D are but you do not necessarily have to find Q^{-1}). Along the way, for each eigenvalue you should explicitly state the algebraic and geometric multiplicity, and give a basis for the eigenspace. (All matrices here are over the real numbers.)

   (a) $A = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

2. In each part below, you are given a linear operator T defined a vector space V. Find the eigenvalues of T, and for each eigenvalue state the algebraic and geometric multiplicity, and find a basis for the eigenspace. Then give a basis β for V for which $[T]_\beta$ is diagonal and write the matrix $[T]_\beta$.

   (a) $T : P_1(\mathbb{C}) \to P_1(\mathbb{C})$ defined by $T(a + bx) = (-b + ax)$. (Note that this is over the complex numbers, not the real numbers.)

   (b) (a) $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ defined by $T(a + bx) = a + 2b + x(-3a - 4b)$.

3. (a) Show that if A is a square matrix, then A and A^t have the same eigenvalues.

   (b) For a common eigenvalue λ, let E_λ and F_λ be the corresponding eigenspaces for A and A^t respectively. Prove that $\dim(E_\lambda) = \dim(F_\lambda)$ and deduce that A is diagonalizable if and only if A^t i s diagonalizable.

   (c) Show by example that E_λ and F_λ need not be the same.

4. Let V be a vector space and let $T : V \longrightarrow V$ be a linear transformation. Let W be a T-invariant subspace of V.

   (a) Suppose that $v_1, \ldots, v_k \in V$ are eigenvectors for T with distinct eigenvalues such that $v_1 + \cdots + v_k \in W$. Show that $v_i \in W$ for each i. [Hint: Use induction on k.]

(b) Show that if V is finite dimensional and T is diagonalizable, then T_W (the restriction of T to W) is also diagonalizable. [Hint: Start by showing that there is a set of eigenvectors for T in W that span W.]

5. Let V be a finite dimensional vector space over a field F and let $T : V \longrightarrow V$ and $U : V \longrightarrow V$ be linear operators. Suppose that $TU = UT$.

   (a) Let λ be an eigenvalue of T and let $E_\lambda = \{v \in V \mid T(v) = \lambda v\}$. Show that $U(E_\lambda) \subseteq E_\lambda$.

   (b) Suppose that $\dim E_\lambda = 1$. Show that every nonzero element of E_λ is an eigenvector for U.

   (c) Suppose that T has n distinct eigenvalues in F where n is the dimension of V. Show that there is an ordered basis γ for V such that $[T]_\gamma^\gamma$ and $[U]_\gamma^\gamma$ are both diagonal matrices.