Homework 9 for S24 Math 235 Due Friday, March 29 at midnight on gradescope.

Follow the instructions on the course homework page to complete this assignment. Please adhere to the honesty policy detailed on the website. **Justify your answers fully.**

1. Show that if A is a 3×3 matrix of rank 1, then there exists a 3×1 matrix B and a 1×3 matrix C such that $A = BC$.

2. (a) Show that the linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(f(x)) = (f(a), f(b), f(c))$ is invertible if and only if the constants a, b, and c are distinct.

   (b) Determine $T^{-1} : \mathbb{R}^3 \to P_2(\mathbb{R})$ if $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ is defined by $T(f(x)) = (f(1), f(0), f(-1))$

3. Find the solution set for the following system. Find a basis for the nullspace of the corresponding homogeneous system. Is this system consistent?

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -2 \\ x_1 + 3x_2 - 2x_3 + 4x_4 &= -1 \\ 2x_1 + 3x_2 + 11x_3 + 2x_4 &= -5 \end{aligned}$$

4. (a) Prove that E is an elementary matrix if and only if E^T is.

   (b) Let $A \in M_{m \times n}(\mathbb{F})$. Show that if B can be obtained from A by an elementary row operation, the B^T can be obtained from A^T by the corresponding elementary column operation. (First briefly explain what *corresponding* must mean.)

5. This question completes the proof that the area of the parallelogram determined by vectors u and v in $\mathbb{R}^2$ is equal to $\left|\det\binom{u}{v}\right|$. Let $\delta : M_{2\times 2}(\mathbb{F}) \to \mathbb{F}$ be a function satisfying the following three properties.

   (a) δ is a linear function of each row of the matrix when the other row is held fixed.

   (b) $\delta\binom{u}{u} = 0$. (That is, if the rows are identical, then δ is zero.)

   (c) $\delta(I_2) = 1$.

   Prove that $\delta(A) = \det(A)$.