Homework 8

Due Friday, March 22, 2024 at 11:59pm on gradescope

Academic honesty expectations: Same as on previous homeworks. We remind you that internet searches are not permitted.

1. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 5 & 0 \end{pmatrix}$.

   (a) Find elementary matrices $E_1, E_2, \dots, E_k$ such that $E_1 E_2 \cdots E_k A = I_3$.

   (b) Determine A^{-1}.

   (c) Write A as a product of elementary matrices.

2. (a) Let A be an $m \times n$ (m rows, n columns) matrix and B be an $n \times p$ matrix. Suppose that $\operatorname{rank}(A) = m$ and $\operatorname{rank}(B) = n$. Find the rank of AB. Justify your answer.

   (b) Prove or give a counter example to the following statement:

   If the $m \times n$ (m rows, n columns) matrix A has rank m, then the system $Ax = b$ is consistent for any choice of b.

   (c) Prove or give a counter example to the following statement: For two $m \times n$ matrices A, B, we must have $\operatorname{rank}(A + B) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

3. Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + 2x_2 + x_3 - x_4 = 0\}$. Find a basis for V containing $(1, -1, 2, 1)$.

4. Let $\beta = \{1, x\}$, let $\gamma = \{1, x + 1\}$. and let $T : P_1(\mathbb{R}) \longrightarrow P_1(\mathbb{R})$ be the linear transformation $T(f) = f' + xf'$. Find $[T]_\gamma^\gamma$ and $[T]_\beta^\beta$. Then find a matrix Q such that $Q^{-1}[T]_\beta^\beta Q = [T]_\gamma^\gamma$.

5. (a) Let $A = \begin{pmatrix} 1 & 1 & 3 & 0 & 2 \\ 2 & 0 & 2 & 1 & 3 \\ 3 & 1 & 5 & -1 & 3 \end{pmatrix}$.

Let us write c_i for the ith column of A. Write down B, the reduced row echelon form of A, by making use of the following facts about A.

(i) $c_3 = c_1 + 2c_2$,

(ii) $c_5 = c_1 + c_2 + c_4$, and

(iii) rank(A)=3.

(b) Suppose the reduced row echelon form A' of a matrix A is given by

$$A' = \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

Determine A if the first, second, and fourth columns of A are

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$