Homework 7 for S24 Math 235 Due Friday, March 8 at midnight on gradescope.

Follow the instructions on the course homework page to complete this assignment. Please adhere to the honesty policy detailed on the website. **Justify your answers fully.**

1. Let U and W be vector spaces. We define the product $U \times W$ to mean the set of ordered pairs (u, w) with $u \in U$ and $w \in W$ with operations $(u_1, w_1) + (u_2, w_2) = (u_1 + u_2, w_1 + w_2)$ and $\lambda(u, w) = (\lambda u, \lambda w)$. It is easy to see that $U \times W$ is a vector space under these operations.

   (a) Show that $\dim(U \times W) = \dim U + \dim W$.

   (b) Now suppose that U and W are both subspaces of a vector space V and let $T : U \times W \longrightarrow V$ be the map sending (u, w) to $u + w$. Show that $\dim N(T) = \dim(U \cap W)$.

2. Recall Q2 on Homework 4. You showed that for subspaces U and W of a vector space V, $\dim(U \oplus W) = \dim(U) + \dim(W)$.

   (a) Prove that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$. [Hint: Use 1(b)]

   (b) Given three subspaces U_1, U_2, and U_3 of a vector space V, we may define $U_1 + U_2 + U_3$ as the set of vectors in V of the form $u_1 + u_2 + u_3$, for $u_i \in U_i$. Provide a counterexample showing that the following formula is not always true: $\dim(U_1 + U_2 + U_3) = \dim(U_1) + \dim(U_2) + \dim(U_3) - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(a, b, c) = (0, a, b)$. Determine $[T]_\gamma$ where $\gamma = \{(1, 0, 1), (1, 3, 0), (0, -1, 0)\}$ is a basis for $\mathbb{R}^3$.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ denote the reflection through the plane $x + y + 3z = 0$. Find the matrix $[T]_\beta$ where β is the standard ordered basis for $\mathbb{R}^3$. (Hint: Start by finding $[U]_\beta$ where $U : \mathbb{R}^3 \to \mathbb{R}^3$ is the reflection through the xy-plane. That is, $U(a, b, c) = (a, b, -c)$. Then find a basis γ for $\mathbb{R}^3$ such that $[U]_\beta = [T]_\gamma$.)

5. Suppose $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is linear. Suppose that $[T]_\gamma = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ where $\gamma = \{1 + x^2, 1 + 3x, -x\}$. Find $T(a + bx + cx^2)$.

6. Consider the bases

$$\alpha = \{(1, 2, 1), (0, 1, 0), (0, -1, 1)\}$$

and

$$\gamma = \{(0, -1, 0), (1, 1, 0), (0, 2, 1)\}$$

for $\mathbb{R}^3$. Determine the change of basis matrix $A = [I_{\mathbb{R}^3}]_\gamma^\alpha$ directly. Then compute it by first finding $[I_{\mathbb{R}^3}]_\gamma^\beta$ and $[I_{\mathbb{R}^3}]_\alpha^\beta$ where β is the standard ordered basis.